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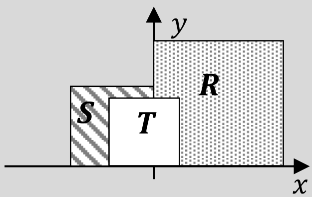
Book **2**.

095. (2022 AMC 10-A)

Let R, S, and T be squares in the coordinate plane, each consisting of all lattice points (points with integer coordinates) that lie on their boundaries or in their interiors. Each square has its bottom side lying along the x-axis. The left side of square R and the right side of square S lie along the y-axis. Square R contains $\frac{9}{4}$ times as many lattice points as square S does. The top two vertices of square T lie within the union of squares R and S, and square T contains exactly $\frac{1}{4}$ of the total number of lattice points in $R \cup S$. Refer to the diagram (not drawn to scale).



The proportion of lattice points in S that lie in the intersection $S \cap T$ is 27 times the proportion of lattice points in R that lie in $R \cap T$.

What is the smallest possible value for the sum of the side lengths of squares R, S, and T?

STEP 1 ›

If the number of lattice points along one side of square S is s, and the number along one side of square R is r, then $r=ks$ for some constant k.

What is the value of k?

CONTINUE

STEP 2 >

The square T contains exactly one-fourth the number of lattice points in the union of $R \cup S$. Assume that the full vertical edge of square S overlaps with that of square R, so those lattice points should be counted only once in the union.

From the result of Step 1, which of the following expressions correctly gives the number of lattice points in T, denoted t^2, in terms of s?

A $t^2 = \frac{9}{4}s^2$

B $t^2 = \frac{13}{16}s^2$

C $t^2 = \frac{13s^2 - 4s}{16}$

D $t^2 = \frac{5s^2 + 4s}{4}$

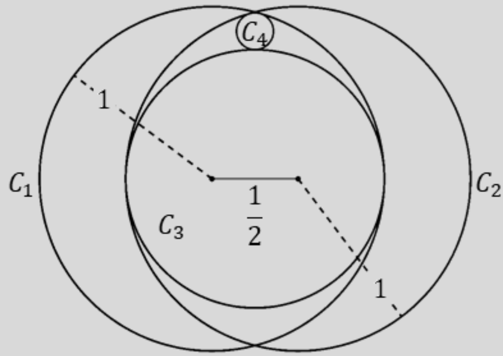
E $t^2 = \frac{1}{4}(r^2 + s^2 + s)$

STEP 3 >

What is the minimum possible value of the sum of the side lengths of squares R, S, and T, assuming all side lengths are positive integers?

099. (2023 AMC 10-A)

Each of the circles C_1 and C_2 has a radius of 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and externally tangent to C_3.



What is the radius of C_4?

STEP 1 >

Let the center of C_1 be P_1 and the center of C_2 be P_2. Let the center of circle C_3 be P_3, and suppose C_3 is the largest possible circle that is internally tangent to both C_1 and C_2.

What is the distance from P_3 to the point of tangency with circle C_2?

CONTINUE

STEP 2 >

Another circle C_4, centered at P_4, is internally tangent to both C_1 and C_2, and externally tangent to C_3. The centers P_1, P_3, and P_4 lie on the same vertical line, and the radius of C_4 is r.

Which of the following correctly gives the distances P_1P_4 and P_3P_4 in terms of r?

A $P_1P_4 = 1 - r,\qquad P_3P_4 = 0.75 + r$

B $P_1P_4 = 1 + r,\qquad P_3P_4 = 0.75 - r$

C $P_1P_4 = 1 - r,\qquad P_3P_4 = 0.75 + r$

D $P_1P_4 = 1 - r,\qquad P_3P_4 = 0.75 - r$

STEP 3 >

Using the same configuration and the result from in Steps 1 and 2, the radius of circle C_4 is $r = \frac{a}{b}$ a in lowest terms.

What is $a+b$?

127. (2024 AMC 10-A)

An equilateral triangle with a height of 36 has one of its sides resting on the line l. A circle of radius 18 is tangent to line l and is externally tangent to the triangle. The area outside both the triangle and the circle, but enclosed by the triangle, the circle, and the line l, can be expressed as $p\sqrt{q} - r\pi$, where p, q, and r are positive integers and q is not divisible by the square of any prime.

What is the value of $p + q + r$?

STEP 1 ›

A circle of radius 18 is tangent to a horizontal line and also externally tangent to one side of an equilateral triangle of height 36.

What is the central angle (in degrees) of the arc formed between the two tangency points?

CONTINUE

STEP 2 ›

In the same configuration, what is the area of $\square ODCE$, formed by the circle's center and its two points of tangency with the triangle and line l?

A $54\sqrt{3}$

B $72\sqrt{3}$

C $108\sqrt{3}$

D $120\sqrt{3}$

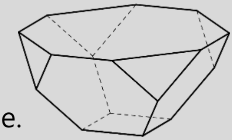
E $124\sqrt{3}$

STEP 3 ›

Given the configuration before, what is the area of the region bounded by the triangle, the circle, and the line, lying outside both the triangle and the circle?
Express your answer in the form $p\sqrt{q} - r\pi$ and find $p + q + r$.

131. (2022 AMC 10-A)

Four regular hexagons of side 2 are attached to a square of side 2 to form a bowl. As illustrated in the figure, the edges of the neighboring hexagons coincide.



What is the area of the octagon formed by connecting the top eight vertices of the four hexagons located on the edge of the bowl?

STEP 1 ›

Each of the four hexagons used to build the bowl in the figure is a regular hexagon with side length 2.

Which of the following correctly represents the number and type of congruent triangles that can be used to tile the interior of one regular hexagon?

A 3 right triangles

B 4 isosceles triangles

C 6 right triangles

D 6 equilateral triangles

E 12 equilateral triangles

CONTINUE

STEP 2 ›

Four regular hexagons with side length 2 are attached to the four sides of a square, also with side length 2, forming a bowl-shaped structure. When the outermost vertices of the hexagons are connected, they form a regular octagon lying flat on the top.

What is the side length of the smallest square that can contain this octagon?

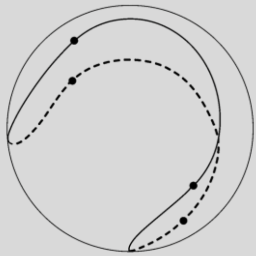
STEP 3 ›

A flat octagon is formed by connecting the outermost vertices of four regular hexagons (each with side length 2) that are attached to each side of a central square (also with side length 2). As established in Step 2, the entire octagon lies within a square of side length 6.This octagon is created by removing four identical right isosceles triangles, one from each corner of the square.

What is the area of this octagon?

132. (2023 AMC 10-B)

Four congruent semicircles are drawn on the surface of a sphere with radius 2, as shown, creating a close curve that divides the surface into two congruent regions. The curve has a total length of $\pi\sqrt{n}$.



What is the value of n?

STEP 1 >

Four points lie on the surface of a sphere with radius 2. Each pair of consecutive points is connected by a congruent semicircular arc on the surface of the sphere, forming a closed curve.

What is the shape of the quadrilateral formed by connecting the points?

A square

B rhombus

C rectangle

D trapezoid

CONTINUE

STEP 2 ›

Based on the result from Step 1, the diameter of each of the four congruent semicircles can be expressed in the form $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime.

What is the value of $a+b$?

STEP 3 ›

A closed curve is formed by four congruent semicircles drawn on the surface of a sphere with radius 2. The curve divides the surface of the sphere into two congruent regions.

If the total length of the curve is expressed as $\pi\sqrt{n}$, what is the value of n?

A 16

B 24

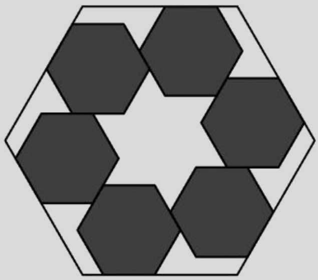
C 32

D 36

E 40

133. (2023 AMC 10-A)

Six unit-side regular hexagons are placed within a larger regular hexagonal boundary. Each hexagonal block is positioned along an inner edge of the frame and aligned with two adjacent blocks, as shown in the below. The distance from any vertex of the outer regular hexagon to the nearest vertex of one of the interior unit-side hexagons, measured along the radial line connecting the center to that vertex, is $\frac{3}{7}$ unit.



What is the area of the region inside the frame not occupied by the blocks?

STEP 1 ›

Six regular hexagonal blocks of side length 1 are placed along the inner edges of a larger regular hexagonal boundary, as shown. The distance from each vertex of the larger hexagon to the nearest vertex of a block is $\frac{3}{7}$.

What is the side length of the larger regular hexagon?

CONTINUE

STEP 2 >

The area of a regular hexagon with side length s can be written in the form $\frac{a\sqrt{b}}{c}s^2$ where a, b, and c are positive integers, and the expression is in simplest form.

What is the value of $a+b+c$?

A 6

B 7

C 8

D 9

E 10

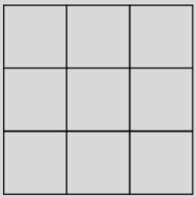
STEP 3 >

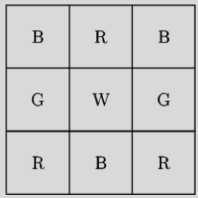
Based on the original problem and the result from Step 1 and Step 2, the area of the region inside the hexagonal frame not occupied by the blocks can be written in the form $\frac{x\sqrt{y}}{z}$ where x, y, and z are positive integers in simplest form.

What is the value of $x+y+z$?

141. (2023 AMC 10-A)

Every square in a 3×3 grid is filled with one of four colors - red, white, blue, or green - in such a way that each 2×2 sub-square includes exactly one square of each color. An example of such a coloring is shown in the diagram to the right below.





How many different ways are there to color the grid?

STEP 1 >

A single 2×2 square is to be colored using the four colors: red, white, blue, and green. Each of the four smaller squares in the 2×2 grid must be colored with a different color.

How many distinct ways are there to color this 2×2 grid?

CONTINUE

STEP 2 ›

Suppose you have already colored the upper-left 2×2 square in a 3×3 grid using all four colors - red, white, blue, and green - such that each square in the 2×2 block has a different color.
Now, consider the overlapping square on the upper-right - that is, the 2×2 block formed by the top-right four cells. Two of its cells (the shared edge squares) have already been colored.

In how many ways can you color the two remaining cells of this second 2×2 square so that it also contains all four colors?

STEP 3 ›

A 3×3 grid is to be colored using the four colors red, white, blue, and green. Each square must be colored with one of the four colors, and the coloring must satisfy the following condition:
Every 2×2 square (there are four such in the grid) must contain all four colors exactly once.

How many such colorings of the entire 3×3 grid are possible?

158. (2022 AMC 10-A)

A sequence of 13 cards labeled 1, 2, 3, ..., 13 is laid out in a single row. You must collect the cards in increasing numerical order, making repeated passes from left to right. In the example below, cards 1 through 3 are collected during the first pass, 4 and 5 during the second, 6 on the third, 7 through 10 on the fourth, and 11 through 13 on the fifth.

$$\boxed{7}\ \boxed{11}\ \boxed{8}\ \boxed{6}\ \boxed{4}\ \boxed{5}\ \boxed{9}\ \boxed{12}\ \boxed{1}\ \boxed{13}\ \boxed{10}\ \boxed{2}\ \boxed{3}$$

For how many of the 13! possible orderings of the cards will the 13 cards be picked up in exactly two passes?

STEP 1 ›

Let

$$S = \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3}.$$

What is the value of S?

A $\binom{6}{3}$

B $\binom{7}{3}$

C $\binom{7}{4}$

D $\binom{6}{3} + 4!$

E $\binom{5}{4} + \binom{6}{4}$

STEP 2 ›

You are given cards labeled with the numbers 1 through 6, arranged in a row. You collect cards by walking from left to right and picking up cards in increasing numerical order. You are allowed to make two passes from left to right through the row.

Which of the following card arrangements cannot be collected in exactly two passes?

A 1 3 2 4 5 6

B 1 2 4 3 5 6

C 1 4 2 5 3 6

D 1 2 3 4 5 6

E 2 1 3 4 5 6

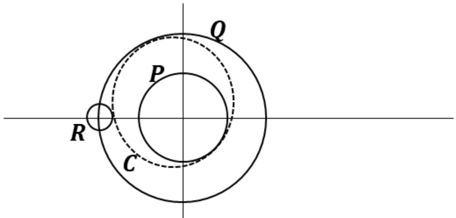
STEP 3 ›

A row of cards numbered 1 through 13 is laid out in some order. You can collect cards by walking from left to right, picking them up in increasing numerical order. You are allowed to walk through the row exactly twice.

How many such arrangements of the 13 cards allow you to collect all of them in exactly two passes?

093. ***Answer and Explanation***

We denote by P the circle that has the equation $x^2 + y^2 = 16$, by Q that $x^2 + y^2 = 36$, and by R that $(x+6)^2 + y^2 = 5$. Also, we denote by $\boldsymbol{C}$ **a circle** that is tangent to P, Q, and R, and by $(\boldsymbol{a}, \boldsymbol{b})$ **the coordinates of circle $\boldsymbol{C}$, and $\boldsymbol{r}$ the radius of this circle**.



From the graphs of circles P, Q, R, we observe that if C is tangent to all of them, then $\boldsymbol{C}$ **must be internally tangent to $\boldsymbol{Q}$**. We have $\boldsymbol{a^2} + \boldsymbol{b^2} = (\boldsymbol{6} - \boldsymbol{r})^{\boldsymbol{2}}$. (1)

We do the following casework analysis in terms of whether $\boldsymbol{C}$ **is externally tangent to $\boldsymbol{P}$ and $\boldsymbol{R}$**.

Case 1: C is externally tangent to P and R.

we have $\boldsymbol{a^2} + \boldsymbol{b^2} = (\boldsymbol{r} + \boldsymbol{4})^{\boldsymbol{2}}$ **(2)**

and $(\boldsymbol{a} + \boldsymbol{6})^{\boldsymbol{2}} + \boldsymbol{b^2} = \left(\boldsymbol{r} + \sqrt{\boldsymbol{5}}\right)^{\boldsymbol{2}}$**.** **(3)**

Taking $(\boldsymbol{2}) - (\boldsymbol{1})$, we get $(\boldsymbol{r} + \boldsymbol{4})^{\boldsymbol{2}} - (\boldsymbol{6} - \boldsymbol{r})^{\boldsymbol{2}} = \boldsymbol{0}$. **Thus,** $\boldsymbol{r} = \boldsymbol{1}$.

Case 2: P is internally tangent to C, R is externally tangent to C.

we have $\boldsymbol{a^2} + \boldsymbol{b^2} = (\boldsymbol{r} - \boldsymbol{4})^{\boldsymbol{2}}$ **(4)**

and $(a+6)^2 + b^2 = \left(r + \sqrt{5}\right)^2$. (5)

Taking $(\boldsymbol{4}) - (\boldsymbol{1})$, we get $(\boldsymbol{r} - \boldsymbol{4})^{\boldsymbol{2}} - (\boldsymbol{6} - \boldsymbol{r})^{\boldsymbol{2}} = \boldsymbol{0}$. Thus, $\boldsymbol{r} = \boldsymbol{5}$.

Case 3: P is externally tangent to C, R is internally tangent to C.

we have $\boldsymbol{a^2} + \boldsymbol{b^2} = (\boldsymbol{r} + \boldsymbol{4})^{\boldsymbol{2}}$ **(6)**

and $(a+6)^2 + b^2 = \left(r - \sqrt{5}\right)^2$. (7)

Taking $(\boldsymbol{6}) - (\boldsymbol{1})$, we get $(\boldsymbol{r} + \boldsymbol{4})^{\boldsymbol{2}} - (\boldsymbol{6} - \boldsymbol{r})^{\boldsymbol{2}} = \boldsymbol{0}$. Thus, $\boldsymbol{r} = \boldsymbol{5}$.

Case 4: P is internally tangent to C, R is internally tangent to C.

we have $\boldsymbol{a^2} + \boldsymbol{b^2} = (\boldsymbol{r} - \boldsymbol{4})^{\boldsymbol{2}}$ **(8**)

and $(a+6)^2 + b^2 = \left(r + \sqrt{5}\right)^2$. (9)

Taking $(\boldsymbol{8}) - (\boldsymbol{1})$, we get $(\boldsymbol{r} - \boldsymbol{4})^{\boldsymbol{2}} - (\boldsymbol{6} - \boldsymbol{r})^{\boldsymbol{2}} = \boldsymbol{0}$. Thus, $\boldsymbol{r} = \boldsymbol{1}$.

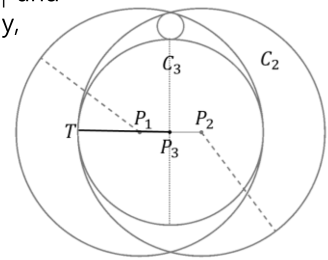
Because the graph is **symmetric with the x-axis**, and for each case above, the solution of b is not 0. Hence, in each case, there are two congruent circles whose centers are symmetric through the x-axis.

So, the **sum of the areas of all the circles in $\boldsymbol{C}$ is**

$$2(1^2\pi + 5^2\pi + 1^2\pi + 5^2\pi) = \boldsymbol{104\pi}.$$

Therefore, the correct answer is **D**.

Since C_3 is internally tangent to both C_1 and C_2, and lies between them symmetrically, the **center P_3 of circle C_3** must lie on the perpendicular bisector of P_1P_2. But more importantly, since the configuration is symmetric and both large circles are identical in radius, the **center of C_3** lies at the **midpoint of P_1 and P_2**.



Thus,

$$\boldsymbol{P_1P_3 = P_2P_3 = \frac{1}{4}}$$

since $P_1P_2=\frac{1}{2}$, and P_3 is at the midpoint.

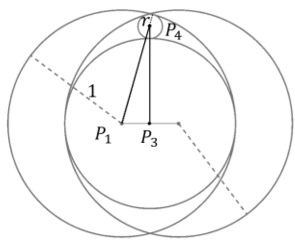
So, the **distance from P_3 to the point of tangency with circle C_2**, or **radius of C_3** is:

$$\boldsymbol{P_3T = P_2T - P_2P_3} \quad \rightarrow \quad \boldsymbol{P_3T} = 1 - \frac{1}{4} = \boldsymbol{\frac{3}{4}}.$$

Therefore, the correct answer is **3/4** or **0.75**.

Step 2.

Let's visualize the centers P_1, P_3, and P_4 on a coordinate grid.
P_1: center of C_1,
P_3: center of C_3, lies between P_1 and P_2,
P_4: center of C_4, which lies below P_1.



From Step 1:

$\boldsymbol{P_1P_3=\frac{1}{4}}$, and the **radius of C_3 is $\frac{3}{4}$**.

We are told:
C_4 is internally tangent to C_1:
→ so distance from P_1 to P_4 is $1-r$: $\quad \boldsymbol{P_1P_4 = 1 - r}$
C_4 is externally tangent to C_3:
→ so distance from P_3 to $P4$ is the sum of their radii:

$$\boldsymbol{P_3P_4 = \frac{3}{4} + r}.$$

Therefore, the correct answer is **C**.

Step 3.

We are given the following distances:

$$P_3P_4 = \frac{3}{4} + r, \qquad P_1P_3 = \frac{1}{4}, \qquad P_1P_4 = 1 - r.$$

Since $\triangle P_1P_3P_4$ is a right triangle with a **right angle at P_3**, we can apply the Pythagorean Theorem:

Now apply Euler's formula:

$$V - E + F = 2 \quad \rightarrow \quad V - 30 + 20 = 2 \quad \rightarrow \quad \boldsymbol{V = 12}.$$

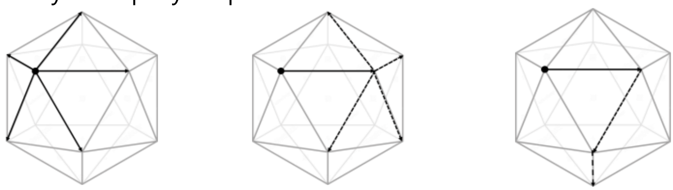
So, a regular icosahedron has **12 vertices**.

Therefore, the correct answer is **12**.

Step 2.

Each vertex is **connected to 5 neighboring vertices** by direct edges. That is, the icosahedron is a 5-regular graph on 12 vertices. This means from any given vertex, **there are 5 vertices at distance 1** (immediate neighbors).

Let's analyze step-by-step:



distance 1: a vertex is connected to its 5 immediate neighbors,
distance 2: from those 5 neighbors, each connects to other vertices (excluding the original vertex),
distance 3: at this point, nearly all vertices are accessible.

In fact, due to the high connectivity of the icosahedron and its symmetry, **any vertex can be reached from any other in at most 3 steps**.

To support this, we can conceptually layer the vertices:

layer 0: starting vertex
layer 1: 5 direct neighbors (distance 1)
layer 2: vertices reachable via 2 edges
layer 3: the vertex farthest from the start.

Thus, the farthest any vertex can be from another is **3 edges** away.

Therefore, the correct answer is **A**.

Step 3.

From Step 1, we learned that the **regular icosahedron has**:

12 vertices, 30 edges, and 20 triangular faces.

From Step 2, we concluded that the **maximum** possible distance between any two vertices **is 3**. Thus, **for any vertex pair A, B** we have: **1≤ $d(A,B)$ ≤3**.

We are **choosing 3 distinct vertices** Q, R, S from **12 total vertices**. So the number of unordered triples is: $\binom{12}{3} = \mathbf{220}$. But the