TABLE OF CONTENTS

Book 1: **ALGEBRA FUNDERMENTALS**

Core Algebraic Concepts

Linear Equations from Verbal Relationships

PROBLEM NO. **003** | DIFFICULTY | 2024 AMC 8 #12

Liam has 110 goldfish in 4 aquariums.

The second aquarium has 1 more goldfish than the first aquarium. The third aquarium has 2 more goldfish than the second aquarium. The fourth aquarium has 3 more goldfish than the third aquarium.

How many goldfish are in the fourth aquarium?

STEP 1 **Concept**

Let x be the number of goldfish in the first aquarium.

Complete the table.

Aquarium	Number of Goldfish
First	x
Second	
Third	
Fourth	

STEP 2 Discovery

Use your answers from Step 1.

What equation represents the total of 110 goldfish in all four aquariums?

STEP 3 Application

Use your equation from Step 2.

How many goldfish are in the fourth aquarium?

Core Algebraic Concepts

Linear Equations with Integer Constraints

PROBLEM NO. 010 | DIFFICULTY | 2022 AMC 8 #13

How many positive integers can correctly complete the statement below?

"One whole number is larger than double of another by _____, and together their sum equals 28."

STEP 1 Concept

Let two positive integers be m and n, where m is larger than n. The problem states that:
"one number is larger than double of another by _d_, and together their sum equals 28."

Which of the following equations correctly expresses the relationship between m, n, and d?

A $m = n + d$

B $m = 2n + d$

C $m = 28 - d - n$

D $m = 28 - n + d$

E $m = 2n - d$

STEP 2 Discovery

Using the result of Step 1,

Which option correctly gives d in terms of n and the condition on n so that d is positive?

A $d = 28 - 2n;\ \ n \leq 14$

B $d = 28 - 3n;\ \ n \leq 9$

C $d = 28 - n;\ \ n \leq 27$

D $d = 3n - 28;\ \ n \geq 10$

E $d = 28 - 3n;\ \ n \leq 10$

STEP 3 Application

How many positive integers can fill the blank d in the statement?

Sequences and Pattern

Pattern-Based Grouping

PROBLEM NO. 026

DIFFICULTY

2026 AMC 8 #01

What is the value of the following expression?

$$2+3-4+5+6-7+8+9-10+11+12-13$$

STEP 1 **Discovery**

The expression can be divided into groups of three numbers.

Complete the table.

Group	Value
$2+3-4$	
$5+6-7$	
$8+9-10$	
$11+12-13$	

STEP 2 **Application**

Use your answers from Step 1.

What is the value of the following expression?

$$2+3-4+5+6-7+8+9-10+11+12-13$$

Sequences and Pattern

Arithmetic Sequences (Nth Term)

PROBLEM NO. 027

DIFFICULTY ■■□□□

2024 AMC 8 #04

> Emma is counting backward by 7s.
> Her first four numbers are 100, 93, 86, and 79.
>
> What is her 13th number?

STEP 1 Discovery

An arithmetic sequence has first term a_1 and common difference d.

Complete the formula for the nth term of the sequence.

$$a_n = a_1 + (\qquad)d$$

STEP 2 Application

Emma is counting backward by 7s. Her first four numbers are 100, 93, 86, and 79.

What is her 13th number?

Sequences and Pattern

Arithmetic Sequences

PROBLEM NO.	DIFFICULTY	
028	■■■□□	2026 AMC 8 #14

Kate chose three equally spaced integers on a number line. The sum of the first and second integers is 40, and the sum of the second and third integers is 80.

What is the sum of all three integers?

STEP 1 **Concept**

Three integers are equally spaced on a number line.

Which statement best describes these integers?

A The three integers have the same value.

B The three integers are consecutive integers.

C The difference between consecutive integers is always the same.

D The sum of the first and third integers equals the middle integer.

STEP 2 Discovery

Three integers are equally spaced on a number line.

Suppose the middle integer is x, and the difference between consecutive integers is y.

Write the three integers in terms of x and y.

STEP 3 Application

The sum of the first and second integers is 40, and the sum of the second and third integers is 80.

ⓐ What is the value of the middle integer?

ⓑ What are the three integers?

ⓒ What is the sum of the three integers?

Sequences and Pattern

Arithmetic Sequences and Consecutive Integer Sums

PROBLEM NO.	DIFFICULTY	
029	■■■■□	2026 AMC 8 #18

The first odd positive integer is n.

Write an expression for the sum of the first four consecutive odd positive integers beginning with n.

STEP 1 **Concept**

A person walks at 2 meters per second, while another person runs at 6 meters per second.

If they move for the same amount of time, what is the ratio of the distance walked to the distance run?

STEP 2 Discovery

The first odd positive integer is n.

Complete the table.?

Number of consecutive odd positive integers	Sum
2	
3	
4	$4n + 12$
5	
6	
7	
8	
9	

STEP 3 Application

Use the result of Step 2 to determine in how many ways 72 can be written as the sum of two or more consecutive odd positive integers listed in increasing order.

Sequences and Pattern

Mutually Exclusive Pattern Events in a Grid

PROBLEM NO.
030

DIFFICULTY

2023 AMC 8 #25

A sequence of fifteen integers $a_1,\ a_2,\ a_3,\ \dots,\ a_{15}$ is placed in increasing order on a number line. These integers are evenly spaced, and they satisfy the conditions

$$1 \le a_1 \le 10, 13 \le a_2 \le 20, \text{ and } 241 \le a_{15} \le 250.$$

What is the sum of digits of a_{14}?

STEP 1 **Concept**

Fifteen integers $a_1,\ a_2,\ \dots,\ a_{15}$ form an arithmetic sequence. They satisfy $13 \le a_2 \le 20$ and $241 \le a_{15} \le 250$.

Using only these two inequalities, which of the following lists all possible integer values of the common difference d?

A 16

B 17

C 18

D 16 or 17

E 17 or 18

STEP 2 Discovery

Using the result of Step 1,
now also use the bounds $1 \leq a_1 \leq 10$ and $241 \leq a_{15} \leq 250$.

Which choice lists all possible values of d that satisfy these new bounds?

A 16

B 17

C 18

D 16 or 17

E 17 or 18

STEP 3 Application

What is the sum of the digits of a_{14} ?

Sequences and Pattern

Multiplicative Recurrence and Exponent Patterns

PROBLEM NO.
036

DIFFICULTY
■■■■■

2023 AMC 8 #22

A sequence is formed from positive integers so that each term after the second equals the product of the two previous terms.

If the sixth term of the sequence is 4000, what is the value of the first term?

STEP 1 **Concept**

Let the first term be a and the second term be b.

Which expression correctly gives the third and fourth terms?

A $T_3 = a + b, \qquad T_4 = a + 2b$

B $T_3 = ab, \qquad T_4 = a^2 b$

C $T_3 = ab, \qquad T_4 = ab^2$

D $T_3 = a^2, \qquad T_4 = b^2$

E $T_3 = a + b, \qquad T_4 = ab$

STEP 2 Discovery

Continuing this pattern, what is the 6th term T_6 in terms of a and b?

A a^3b^3

B a^2b^4

C a^3b^4

D a^3b^5

E a^4b^5

STEP 3 Application

Suppose T_6=4000.

What is the value of the first term a?

with x=16:

l_1: $y = \frac{1}{3}x \rightarrow \boldsymbol{y} = \frac{1}{3}(\mathbf{16}) = \frac{\mathbf{16}}{\mathbf{3}} > \mathbf{5} \rightarrow$ **(out side)**

l_2: $y = -\frac{1}{2}x + 10 \rightarrow \boldsymbol{y} = -\frac{1}{2}(\mathbf{16}) + 10 = \mathbf{2} < \mathbf{3} \rightarrow$ (**out side**)

② intersections with the **horizontal edges**

with $\boldsymbol{y}$=**5**:

l_1: $y = \frac{1}{3}x \rightarrow \mathbf{5} = \frac{1}{3}x \rightarrow \boldsymbol{x} = \mathbf{15} \rightarrow$ (**on boundary**)

l_2: $y = -\frac{1}{2}x + 10 \rightarrow \mathbf{5} = -\frac{1}{2}x + 10 \rightarrow \boldsymbol{x} = \mathbf{10} \rightarrow$ (**out side**)

with $\boldsymbol{y}$=**3**:

l_1: $y = \frac{1}{3}x \rightarrow \mathbf{3} = \frac{1}{3}x \rightarrow \boldsymbol{x} = \mathbf{9} \rightarrow$ (**out side**)

l_2: $y = -\frac{1}{2}x + 10 \rightarrow \mathbf{3} = -\frac{1}{2}x + 10 \rightarrow \boldsymbol{x} = \mathbf{14} \rightarrow$ (**out side**)

The only boundary point hit by either line is **(15, 5)**, and the **total is 1**.

Therefore, the correct answer is **1** or **one**.

014. *Answer and Explanation*

Step 1.

Because **every row** has the **same total**, we can compute the common sum by adding the entries in any complete row. The first row is −2, 9, 5. Thus, **(−2)+9+5=12**.

Since **all rows** (and **all columns**) share this total, the common sum must be **12**.

Therefore, the correct answer is **12**.

Step 2.

From Step 1, the common row/column sum is 12. Let the two blank numbers in the second row be $\boldsymbol{a}$ **and** $\boldsymbol{b}$. Since the right entry is −1, the second-row sum gives

$$\boldsymbol{a} + \boldsymbol{b} + (-\mathbf{1}) = \mathbf{12} \quad \rightarrow \quad \boldsymbol{a} + \boldsymbol{b} = \mathbf{13}.$$

Thus, the sum of the two blanks in the second row is 13.

Therefore, the correct answer is **13**.

Step 3.

From Step 2 we know $a+b=13$.
If both a and b were ≤6, their sum would be at most 12, which is impossible. Hence **at least one of a, b is ≥7**, so the largest among the second-row blanks is at least 7.

By the problem condition, **x is greater than all three of the other missing numbers.** Since one of those three is at least 7, we must have $x>7 \rightarrow$ **$x\geq 8$**.

To show **$x=8$** is achievable (so it's the minimum), we can complete the grid so that each row/column sums to 12 and the other **three blanks are all <8**.
Each row/column sums to 12, and the three other missing numbers are 6, 7, −4 all less than $x=8$.

−2	9	5
6	**7**	−1
8	**−4**	8

Thus the **smallest possible value of x is 8**.

Therefore, the correct answer is **8**.

015. *Answer and Explanation*

The sum of the numbers in **each row is 9+(-3)+9=15**, and the sum of the numbers in **each column is 9+7+(-1)=15**. To find the smallest possible value of X, we need to determine the value of the three missing numbers that are less than X.

$$\boldsymbol{a + X + (-1) = 15 \rightarrow a = 16 - X}$$
$$\boldsymbol{b + X + (-3) = 15 \rightarrow b = 18 - X}$$
$$\boldsymbol{c + a + 9 = 15 \rightarrow c + 16 - X + 9 = 15 \rightarrow c = -10 + X}$$

-1	X	a
7	b	c
9	-3	9

-1	X	$16-X$
7	$18-X$	$-10+X$
9	-3	9

-1	10	6
7	8	0
9	-3	9

Other three numbers are expressed as **16-X, 18-X, and -10+X** and these three numbers are less than X.

$$\boldsymbol{X > 16 - X \rightarrow X > 8}$$
$$\boldsymbol{X > 18 - X \rightarrow X > 9}$$
$$\boldsymbol{X > -10 + X \rightarrow X \text{ is all integers.}}$$

So, **X>9 and X is an integer** and the **smallest possible value of X is 10**.

Therefore, the correct answer is **10**.

Step 3.

ⓐ The middle integer appears in both sums, so combining the two equations isolates the middle integer.

The three integers are $x-y$, x, $x+y$.
And from the given information, $(\boldsymbol{x}-\boldsymbol{y})+\boldsymbol{x}=\mathbf{40}$, $\boldsymbol{x}+(\boldsymbol{x}+\boldsymbol{y})=\mathbf{80}$.
Add the two equations:

$$x-y+x=40 \quad \rightarrow \quad \mathbf{2}\boldsymbol{x}-\boldsymbol{y}=\mathbf{40} \quad (1)$$
$$x+x+y=80 \quad \rightarrow \quad \mathbf{2}\boldsymbol{x}+\boldsymbol{y}=\mathbf{80} \quad (2)$$

$(1)+(2)$:

$$\mathbf{2}\boldsymbol{x}-\boldsymbol{y}+\mathbf{2}\boldsymbol{x}+\boldsymbol{y}=\mathbf{40}+\mathbf{80} \quad \rightarrow \quad 4x=120 \quad \rightarrow \quad \boldsymbol{x}=\mathbf{30}.$$

So,
the **middle integer is 30**.

Therefore, the correct answer is **30**.

ⓑ Once the middle integer is known, only the common difference remains to determine the other two integers.

After finding the middle integer, only the common difference needs to be determined. **Substitute $\boldsymbol{x}=\mathbf{30}$ into $\mathbf{2}\boldsymbol{x}-\boldsymbol{y}=\mathbf{40}$**.
So,

$$2x-y=40 \quad \rightarrow \quad 2(\mathbf{30})-y=40 \quad \rightarrow \quad \boldsymbol{y}=\mathbf{20}.$$

Now,

$$\boldsymbol{x}-\boldsymbol{y}=30-20=\mathbf{10}, \qquad \boldsymbol{x}+\boldsymbol{y}=30+20=\mathbf{50}.$$

Therefore, the correct answer is **10, 30, 50**.

ⓒ The **sum of the three integers** is:

$$10+30+50=\mathbf{90}.$$

Therefore, the correct answer is **90**.

029. *Answer and Explanation*

Step 1.

Consecutive odd positive integers always **increase by 2**.

Odd positive integers **differ by 2**, so **each new term** is obtained by **adding 2** to the previous term.
If the **first odd positive integer** is $\boldsymbol{n}$, then the next three consecutive odd positive integers are: $\boldsymbol{n}+\mathbf{2}$, $\boldsymbol{n}+\mathbf{4}$, and $\boldsymbol{n}+\mathbf{6}$.

The **sum of the four consecutive odd positive integers** beginning with n is:

$$\boldsymbol{n}+(\boldsymbol{n}+\mathbf{2})+(\boldsymbol{n}+\mathbf{4})+(\boldsymbol{n}+\mathbf{6})=\mathbf{4}\boldsymbol{n}+\mathbf{12}.$$

Therefore, the correct answer is $\mathbf{4}\boldsymbol{n}+\mathbf{12}$.

Step 2.

When several expressions are related, **look for a pattern** instead of simplifying every expression separately.

Begin by simplifying only the first few expressions.
For **2** consecutive odd positive integers,

$$n + (n + 2) = \mathbf{2n + 2}.$$

For **3** consecutive odd positive integers,

$$n + (n + 2) + (n + 4) = \mathbf{3n + 6}.$$

For 4 consecutive odd positive integers, from Step 1 result, $\mathbf{4n + 12}$.

Now compare these expressions carefully.
What do you notice about the **coefficient of n**?
How does the **constant term change**?
A clear pattern appears.
The **coefficient of n** is always equal to the **number of consecutive odd positive integers**. The constant term increases by the next even number each time. **2**, **6**, **12**, **20**, **30**, **42**, **56**, **72** since the **increases are +4, +6, +8, +10, +12, +14, +16**.

Use these patterns to complete the remaining rows of the table without simplifying every expression individually.

Therefore, the correct answer is:

Number of consecutive odd positive integers	Sum
2	$\mathbf{2n+2}$
3	$\mathbf{3n+6}$
4	$4n+12$
5	$\mathbf{5n+20}$
6	$\mathbf{6n+30}$
7	$\mathbf{7n+42}$
8	$\mathbf{8n+56}$
9	$\mathbf{9n+72}$

Step 3.

Use the pattern discovered in Step 2 to **test every possible number of consecutive odd positive integers**.

From Step 2, each possible number of consecutive odd positive integers has a **corresponding expression for its sum**. By setting each expression **equal to 72**, we obtain one equation for each possible number of terms. Since each equation has only one possible value of n, **checking every equation guarantees that every possible case is tested without missing any valid representation**.

036. *Answer and Explanation*

Step 1.

The rule says:
each term after the second equals the **product of the previous two**. In symbols, for $n \geq 3$,

$$\boldsymbol{T_n = T_{n-1} \cdot T_{n-2}}.$$

We are given $\boldsymbol{T_1} = \boldsymbol{a}$ and $\boldsymbol{T_2} = \boldsymbol{b}$.
Then the third term is $\boldsymbol{T_3} = \boldsymbol{T_2} \times \boldsymbol{T_1} = \boldsymbol{ab}$,
and the fourth term multiplies the previous two terms T_3 and T_2:

$$\boldsymbol{T_4} = \boldsymbol{T_3} \times \boldsymbol{T_1} = (\boldsymbol{ab})\boldsymbol{b} = \boldsymbol{ab^2}.$$

Therefore, the correct answer is **C**.

Step 2.

Find T_5:

$$\boldsymbol{T_5} = T_4 \times T_3 = (ab^2) \cdot (ab) = \boldsymbol{a^2 b^3}.$$

Find T_6:

$$\boldsymbol{T_6} = T_5 \times T_4 = \left(\boldsymbol{a^2 b^3}\right) \cdot \left(\boldsymbol{ab^2}\right) = \boldsymbol{a^3 b^5}.$$

So,
the **6th term** in terms of a and b is: $\boldsymbol{a^3 b^5}$.

Therefore, the correct answer is **D**.

Step 3.

① prime-factorize 4000

$$\mathbf{4000} = 4 \times 1000 = 2^2 \times (2^3 \times 5^3) = \mathbf{2^5} \times \mathbf{5^3}.$$

② match exponents
Since a and b are positive integers and $\boldsymbol{a^3 b^5} = \mathbf{2^5} \cdot \mathbf{5^3}$, the only way to distribute the prime powers consistently is to let the base of $\boldsymbol{a}$ account for the prime whose **total exponent is** a multiple of **3**, and the base of $\boldsymbol{b}$ account for the prime whose **total exponent is** a multiple of **5**.
The factor $\mathbf{5^3}$ is a perfect cube, so it must come from $\boldsymbol{a^3}$.
Hence $\boldsymbol{a}$=**5**.
The factor $\mathbf{2^5}$ is a perfect fifth power, so it must come from $\boldsymbol{b^5}$. Hence $\boldsymbol{b}$=**2**.

The **first term is** $\boldsymbol{a}$=**5**.

Therefore, the correct answer is **5**.