TABLE OF CONTENTS

Book 2: **RATIOS, RATES and NUMBER THEORY**

Ratios and Proportions

Ratio Equations with Variable Expressions

PROBLEM NO. **044** | DIFFICULTY ■■■□□ | 2024 AMC 8 #21

A group of chameleons is living in a tree. A chameleon becomes green in the shade and yellow in the sunlight. he group started with green and yellow chameleons in a 3:1 ratio. Next, 3 green chameleons went to the sunny side, and 5 yellow ones went to the shady side. Now, the ratio of green to yellow chameleons is 4:1.

What is the difference between the number of green chameleons and the number of yellow chameleons now?

STEP 1 Concept

The ratio of green chameleons to yellow chameleons is $3:1$.

If there are y yellow chameleons,
what expression represents the number of green chameleons?

STEP 2 Discovery

Three green chameleons move to the sunny side, and five yellow chameleons move to the shady side.

ⓐ Express the number of green chameleons after these changes in terms of y.

ⓑ Express the number of yellow chameleons after these changes in terms of y.

STEP 3 Application

After the changes, the ratio of green chameleons to yellow chameleons is $4:1$.

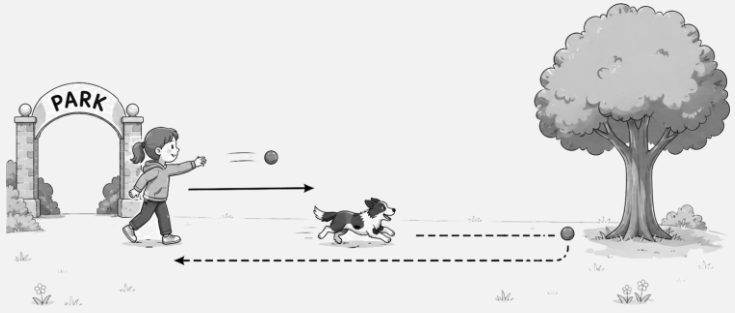
Using your answers from Step 2,
what is the difference between the number of green chameleons and the number of yellow chameleons?

Rates and Percent

Equal-Time Distance Ratios

PROBLEM NO. **054** | DIFFICULTY ■■■■□ | 2026 AMC 8 #19

Nora is walking with her dog, Max. When they reach the entrance to a park, Nora throws a ball straight ahead along the path and continues walking at a steady pace. Max sprints toward the ball, where it comes to rest beside a tree. As soon as he reaches the ball, he brings it back to Nora. Max runs five times as fast as Nora walks.



What fraction of the distance from the park entrance to the tree does Nora cover by the time Max brings her the ball?

STEP 1 Concept

A person walks at 2 meters per second, while another person runs at 6 meters per second.

If they move for the same amount of time, what is the ratio of the distance walked to the distance run?

STEP 2 Discovery

The runner turns around immediately after reaching the marker and later meets the walker.

What can you conclude about the amount of time each person has been moving when they meet?

STEP 3 Application

Now return to the original problem.
Let d be the distance from the park entrance to the tree, and let x be the distance Nora walks before the dog returns.

ⓐ What expression represents the total distance the dog runs?

ⓑ Using the distances traveled by Nora and the dog, write one equation relating x and d.

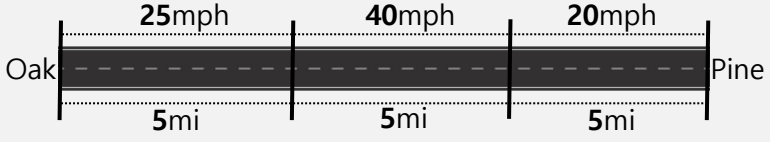
ⓒ Based on your equation, what fraction of the distance from the park entrance to the tree does Nora walk?

Rates and Percent

Piecewise Speed and Meeting Distance

PROBLEM NO. **061** | DIFFICULTY ■■■■□ | 2024 AMC 8 #19

Two towns, Oak and Pine, are connected by a straight road that is 15 miles long. Traveling from Oak to Pine, the speed limit changes every 5 miles: 25 mph, 40 mph, and 20 mph. Two drivers, Emma and Lily, start at the same time, with Emma leaving Oak and Lily leaving Pine. They drive toward each other, traveling exactly at the posted speed limit in each section of the road.



How far from Oak, in miles, do the two drivers meet?

STEP 1 **Concept**

A car travels 20 miles at a constant speed of 40 mph.

How many hours does the trip take?

STEP 2 Discovery

Two road sections are each 5 miles long.

Complete the table.

Section	Speed	Time
First	25 mph	( )
Second	40 mph	( )

STEP 3 Application

A 15-mile road is divided into three 5-mile sections with speed limits of 25 mph, 40 mph, and 20 mph. Two drivers start at the same time from opposite ends of the road and always travel at the posted speed limit.

ⓐ How many hours does it take the driver starting from the left end to reach the 10-mile mark?

ⓑ After that same amount of time, how far is the other driver from the left end of the road?

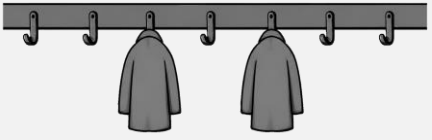
ⓒ How far from the left end do the two drivers meet?

Integers and Divisibility

Equal-Spacing Arrangement Using Divisors

PROBLEM NO. **066** | DIFFICULTY ■■■■■ | 2024 AMC 8 #22

A classroom has a row of 35 coat hooks. Mia wants to arrange the coats so that they are evenly spaced, with the same number of unused hooks before the first coat, after the last coat, and between each pair of consecutive coats. Suppose there is at least 1 coat and at least 1 unused hook.



How many different numbers of coats satisfy Mia's arrangement?

STEP 1 Concept

A row has 4 coats. The unused hooks form equal spaces before the first coat, between consecutive coats, and after the last coat.

How many equal spaces are there?

STEP 2 **Discovery**

A row has 5 coats. Each equal space contains 3 unused hooks.

How many hooks are there altogether?

STEP 3 **Application**

A classroom has a row of 35 coat hooks. Mia arranges some coats so that the same number of unused hooks appears before the first coat, after the last coat, and between each pair of consecutive coats. Suppose there is at least 1 coat and at least 1 unused hook.

ⓐ If there are c coats, how many equal spaces are there?

ⓑ Generalize your result from Step 2 to write an equation for the total number of hooks when there are c coats and each equal space contains h unused hooks.

ⓒ Hence, how many different numbers of coats satisfy Mia's arrangement?

Integers and Divisibility

Prime Factorization and Divisor Counting

PROBLEM NO. **068** | DIFFICULTY ■■■□□ | 2020 AMC 8 #17

Among the positive factors of 2020, how many of them has more than 3 factors?

(For example, the number 12 has 6 factors: 1,2,3,4,6, and 12.)

STEP 1 **Concept**

How many positive divisors does 2020 have?

A 8

B 10

C 12

D 14

E 16

STEP 2 **Discovery**

Among the positive divisors of 2020,
how many have at most 3 positive divisors?

A 4

B 5

C 6

D 7

E 8

STEP 3 **Application**

How many positive divisors of 2020 have more than 3 positive divisors?

Integers and Divisibility

Place Value and Repeated Number

PROBLEM NO. **069** | DIFFICULTY ■■■■□ | 2024 AMC 8 #15

Let the letters R, U, N, D, O, G represent six different digits. Suppose $RUNRUN$ is the largest number that satisfies the equation

$$8 \cdot RUNRUN = DOGDOG.$$

What is the value of $RUN+DOG$?

STEP 1 **Concept**

ⓐ Write the repeated number as a product.

$$RUNRUN = 1001 \times \_\_\_\_\_$$

ⓑ Write the repeated number as a product.

$$DOGDOG = 1001 \times \_\_\_\_\_$$

STEP 2 Discovery

ⓐ Use your answers from Step 1.

Rewrite $8 \times RUNRUN = DOGDOG$ as an equation involving only the three-digit numbers RUN and DOG.

ⓑ Since DOG is a three-digit number, what range of values can RUN have?

$$\leq RUN \leq$$

ⓒ Before checking whether all six letters represent different digits, what is the greatest possible value of RUN?

STEP 3 Application

Use your answers from Steps 1 and 2.

ⓐ Find the greatest value of RUN that satisfies the condition that R, U, N, D, O, and G represent six different digits.

ⓑ Find the greatest possible value of $RUN + DOG$.

Step 2.

We are told there are **250 catfish** in the pond. The equation for the fraction is:

$$\frac{250}{\text{total number of fish}} = \frac{1}{6} \quad \rightarrow \quad \textbf{total number of fish} = 250 \times 6 = \mathbf{1500}.$$

Therefore, the correct answer is **1500**.

039. ***Solution & Answer***

Let's denote **the number of apples Diana received from each friend** as '$\boldsymbol{d}$'.

Assuming **Alex's apples as '$\boldsymbol{a}$'**, **Becky's apples as '$\boldsymbol{b}$'**, and **Celina's apples as '$\boldsymbol{c}$'**, we can set up equations based on the information provided.

Alex decided **to give** Diana **one-third of his apples**, which can be represented as;

$$d = \frac{1}{3}a \quad \rightarrow \quad \boldsymbol{a = 3d}.$$

Similarly, **Becky gave** Diana **one-fourth of her apples**, leading to;

$$d = \frac{1}{4}b \quad \rightarrow \quad \boldsymbol{b = 4d}.$$

And Celina shared one-fifth of her apples with Diana, leading to;

$$d = \frac{1}{5}c \quad \rightarrow \quad \boldsymbol{c = 5d}.$$

To figure out what fraction of the **total apples Diana has now**, we add the apples she received from each friend ($\boldsymbol{3d}$) and divide that by the **total number of apples** ($\boldsymbol{3d + 4d + 5d}$), simplifying to;

$$\frac{\boldsymbol{d + d + d}}{\boldsymbol{a + b + c}} = \frac{\boldsymbol{3d}}{\boldsymbol{3d + 4d + 5d}} = \frac{3d}{12d} = \boldsymbol{\frac{1}{4}}.$$

Therefore, Diana now possesses $\boldsymbol{\frac{1}{4}}$ (one-fourth) of the group's total apples.

040. ***Answer and Explanation***

Let's denote the number of blue, green, red, and white books as $\boldsymbol{b}$, $\boldsymbol{g}$, $\boldsymbol{r}$, and $\boldsymbol{w}$ respectively.

According to the given ratios, we can express g, r, and w in terms of b as follows:

$$b:g = 1:3 \rightarrow \frac{b}{g}=\frac{1}{3} \rightarrow \boldsymbol{g = 3b}$$

$$g:r = 2:5 \rightarrow \frac{g}{r}=\frac{2}{5} \rightarrow 2r = 5g \rightarrow \boldsymbol{r}=\left(\frac{5}{2}\right)g=\left(\frac{5}{2}\right)(3b)=\boldsymbol{\frac{15}{2}b}$$

$$r:w = 4:5 \rightarrow \frac{r}{w}=\frac{4}{5} \rightarrow 4w = 5r \rightarrow \boldsymbol{w}=\frac{5}{4}r=\frac{5}{4}\left(\frac{15}{2}\right)b=\boldsymbol{\frac{75}{8}b}$$

The total number of books in the bookstore is:

$$b+g+r+w = b+3b+\frac{15}{2}b+\frac{75}{8}b=\boldsymbol{\frac{167}{8}b}$$

Since the number of books must be a positive integer, $\boldsymbol{\frac{167}{8}b}$ must **be a positive integer**.

Let's denote this integer as b,

$$\boldsymbol{b = 8k}\text{ (k is some positive integer)}$$

$$\frac{167}{8}b=\frac{167}{8}(8k)=\boldsymbol{167k}$$

Given that the bookstore contains **fewer than 200 books**, the only possible value for $\boldsymbol{k}$ **is 1**, which makes the total number of books **167**.

Therefore, there are **167** books in the Bookstore.

041. ***Answer and Explanation***

Step 1.

Multiply the **population of the starting city by the fraction** on the arrow.

The arrow from P to Q is labeled $\frac{1}{4}$. Since **100 people live in P**,

$$\boldsymbol{100\times\frac{1}{4}=25}.$$

So **25 people who live in P work in Q**.

Therefore, the correct answer is **25**.

048. *Answer and Explanation*

Step 1.

Add fractions that represent parts of the same whole.

Both fractions describe parts of the same battery, so they can be **added to find the total battery used**.

The first trip **used** $\mathbf{\frac{1}{2}}$ of the battery, and the second trip **used** $\mathbf{\frac{3}{10}}$.

Add the fractions.

$$\mathbf{\frac{1}{2}}+\mathbf{\frac{3}{10}}=\frac{5}{10}+\frac{3}{10}=\frac{8}{10}=\mathbf{\frac{4}{5}}.$$

Therefore, the correct answer is $\mathbf{\frac{4}{5}}$.

Step 2.

Use the known fraction of the whole to find the whole amount.

The **80 miles represent only** $\mathbf{\frac{4}{5}}$ **of the bike's full travel distance**, **not the entire distance**.

From Step 1, $\frac{4}{5}$ of the battery was used.

Let the **full travel distance be** $\boldsymbol{x}$ **miles**.

Since $\mathbf{\frac{4}{5}}$ **of the full distance is 80 miles**, write the equation and solve for x:

$$\mathbf{\frac{4}{5}}\boldsymbol{x}=\mathbf{80}\ \ \rightarrow\ \ \boldsymbol{x}=80\times\frac{5}{4}=\mathbf{100}.$$

Therefore, the correct answer is **100** miles.

049. *Answer and Explanation*

Step 1.

"**18% of** $\boldsymbol{m}$ **equals 24% of** $\boldsymbol{n}$" means $\mathbf{0.18}\boldsymbol{m}=\mathbf{0.24}\boldsymbol{n}$.

Divide both sides by $0.24m$ (assuming $m\neq0$):

$$0.18m=0.24n\ \ \rightarrow\ \ \frac{0.18m}{\mathbf{0.24}\boldsymbol{m}}=\frac{0.24n}{\mathbf{0.24}\boldsymbol{m}}\ \ \rightarrow\ \ \frac{18}{24}=\frac{n}{m}$$

$$\rightarrow\ \ \mathbf{\frac{n}{m}}=\mathbf{\frac{3}{4}}$$

Therefore, the correct answer is **B**.

Step 2.
Solve for the ratio and convert to percent:

$$\frac{3}{4} \rightarrow \frac{3}{4} \times \mathbf{100} = \mathbf{75}\%.$$

Therefore, the correct answer is **75%**.

Step 3.
Set up the equation.

$$\mathbf{0.14}\boldsymbol{a} = \mathbf{0.35}\boldsymbol{b}.$$

Isolate $\frac{b}{a}$ and convert to percent.

$$0.35b = 0.14a \rightarrow \frac{0.35b}{\mathbf{0.35}\boldsymbol{a}} = \frac{0.14a}{\mathbf{0.35}\boldsymbol{a}} \rightarrow \frac{\boldsymbol{b}}{\boldsymbol{a}} = \frac{14}{35} = \frac{\mathbf{2}}{\mathbf{5}}.$$

Thus,
b is $\frac{2}{5}$ of a, which is:

$$\frac{2}{5} \rightarrow \frac{2}{5} \times \mathbf{100} = \mathbf{40}\%.$$

Therefore, the correct answer is **B**.

050. *Answer and Explanation*

Step 1.

A percentage increase or decrease is always **calculated from the current amount**. So, when there are multiple percentage changes, you must first find how much remains, and then use that remaining amount as the new base for the next calculation.

A **30% decrease means 30% is taken away from the original amount**. Since the original amount is 100%, the remaining amount is 100%−30%=70%.
So, **70% of the original amount** remains after the decrease.

Therefore, the correct answer is **70%**.

Step 2.

From Step 1, **70% of the original amount remains**.
This remaining **70% becomes the new base** for the next percentage change. The account then **increases by 60% of the current amount, not 60% of the original amount**.

First, find the increase: **0.6×70%=42%**.
Then add the increase to the remaining amount:

70%+42%=112%.

So, the **final amount is 112% of the original amount**.

Therefore, the correct answer is **112%**.

051. *Answer and Explanation*

Step 1.

The CO_2 level **increases by the same amount every year**. So, **multiplying** the yearly increase by the number of years gives the total increase over the entire period.

From January 1985 to January 2035, the **number of years is**:

$$2035 - 1985 = \mathbf{50}.$$

The **total increase** is: $\mathbf{50} \times \mathbf{1.515} = \mathbf{75.75}$ ppm.
Rounded to the nearest whole number, $75.75 \approx \mathbf{76}$.

Therefore, the correct answer is ⓐ **50** and ⓑ **76 ppm**.

Step 2.

The **initial CO_2 level was 338 ppm**, and the **exact total increase** from Step 1 was **75.75 ppm**. **338+75.75=413.75**.

Rounded to the nearest whole number, $\mathbf{413.75} \approx \mathbf{414}$.
The **expected CO_2 level in January 2035 is 414 ppm**.

Therefore, the correct answer is **414 ppm**.

052. *Answer and Explanation*

Step 1.

Find the driving time by **dividing the distance by the average driving speed**.

The relationship between distance, speed, and time is:

$$\textbf{Time} = \frac{\textbf{Distance}}{\textbf{Speed}}.$$

The driver travels **180 km** at an average driving speed of **60 km per hour**.
So,

$$\textbf{Driving Time} = \frac{\mathbf{180}}{\mathbf{60}} = \mathbf{3}.$$

Therefore, the correct answer is **3 hours**.

Step 2.

Average driving speed is calculated using only the time spent driving.

The **total trip time** includes **both the driving time** and **the rest break**. Since the average speed is based only on the driving time, first find the driving time. Then **subtract it from the total trip time to find the length of the rest break**.

056. ***Answer and Explanation***

Step 1.
The average speed of an object is calculated as:

$$\textbf{average speed} = \frac{\textbf{total distance traveled}}{\textbf{total time taken}}.$$

So,

$$\frac{\mathbf{120,000,000}}{\mathbf{4,000}} = \frac{120{,}000}{4} = \mathbf{30,000\ mph}.$$

The spacecraft's average speed is about **30,000 miles per hour**.

Therefore, the correct answer is **C**.

Step 2.
We want hours:
convert months → days → hours

$$\textbf{6.5 months} \times \frac{\textbf{30 days}}{\textbf{1 month}} \times \frac{\textbf{24 hours}}{\textbf{1 day}} = \mathbf{4680\ hours}.$$

4680 is **closest to 4700** among the options.

Therefore, the correct answer is **B**.

Step 3.
Use the definition of average speed

$$\textbf{average speed} = \frac{\textbf{total distance}}{\textbf{total time}} = \frac{292{,}526{,}838}{4700}.$$

So,

$$\frac{292{,}526{,}838}{4700} \approx \frac{\mathbf{300,000,000}}{\mathbf{5000}} \approx \mathbf{60,000\ mph}.$$

Therefore, the correct answer is **D**.

057. ***Answer and Explanation***

Step 1.

① Compute the distance at 10 AM:
time driven: 8 AM → 10 AM= **2 hours**,
speed: **45 mph** (constant),
distance formula: distance=speed×time=**45×2=90miles**.
So at 10 AM, the car is **90 miles from** the apartment.

② Determine the graph from 10 AM to 1 PM.
From 10 AM to 1 PM, the car is **parked (no movement)**. When **distance doesn't change over time**, the distance–time graph is a horizontal line segment (**slope =0**).

Therefore, the correct answer is **B**.

Step 2.

From Step 1, we already know:
8 AM → 10 AM: the car **increases** its distance **to 90** miles.
10 AM → 1 PM: the car stays parked at 90 miles
(**flat horizontal segment**).

At 1 PM, Evan drives home at 60 mph:
distance back = 90 miles,
time required = $\frac{\textbf{90 miles}}{\textbf{60 mph}} = \textbf{1.5 hours}$.

So, Evan arrives home at **2:30 PM**. The graph must show the distance decreasing linearly **from 90 miles at 1 PM to 0 miles at 2:30 PM**. This segment is a straight downward line, and steeper than the outbound line because the return speed is greater.

Therefore, the correct answer is **E**.

058. ***Answer and Explanation***

Step 1.

Read arrival times from the graph.
Thomas (solid line): arrives **at 20 minutes**.
Selina (dashed line): arrives **at 30 minutes**.

Convert minutes to hours.

$$\textbf{20 minutes} = \frac{20}{60} = \frac{\mathbf{1}}{\mathbf{3}}\ \textbf{hour}, \quad \textbf{30 minutes} = \frac{30}{60} = \frac{\mathbf{1}}{\mathbf{2}}\ \textbf{hour}.$$

Therefore, the correct answer is **B**.

Step 2.

Apply the definition of average speed.

$$\textbf{average speed} = \frac{\textbf{distance}}{\textbf{time}}.$$

Thomas:

$$\boldsymbol{v_T} = \frac{\mathbf{5}}{\frac{\mathbf{1}}{\mathbf{3}}} = 15\ mph.$$

Selina:

$$\boldsymbol{v_S} = \frac{\mathbf{5}}{\frac{\mathbf{1}}{\mathbf{2}}} = 10\ mph.$$

Find the difference in speeds.

$$|v_T - v_S| = |\mathbf{15 - 10}| = \mathbf{5}\ \boldsymbol{mph}.$$

Therefore, the correct answer is **5**.