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Book 3: **GEOMETRY**

Planar Geometry Properties

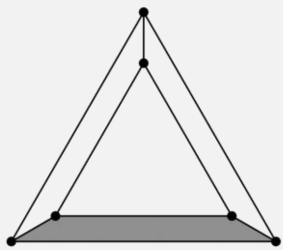
Similar Figures and Area Ratios

PROBLEM NO.
075

DIFFICULTY
■■■□□

2023 AMC 8 #19

Inside a larger equilateral triangle, a smaller equilateral triangle is placed so that the space between them is divided into three equal trapezoids, as shown below. The side length of the smaller triangle is $\frac{2}{3}$ of the side length of the larger triangle.



What is the ratio of the area of one trapezoid to the area of the smaller triangle?

STEP 1 **Concept**

What is the ratio of the area of the larger triangle to the area of the smaller triangle?

STEP 2 Discovery

Using the same figure and information as in Step 1, find the ratio of the area of the region between the two triangles (the ring made of three trapezoids) to the area of the smaller triangle.

A $\frac{1}{3}$

B $\frac{1}{4}$

C $\frac{5}{4}$

D $\frac{5}{12}$

E $\frac{9}{4}$

STEP 3 Application

Since this ring is divided into three congruent trapezoids, what is the ratio of the area of one trapezoid to the area of the smaller triangle?

Planar Geometry Properties

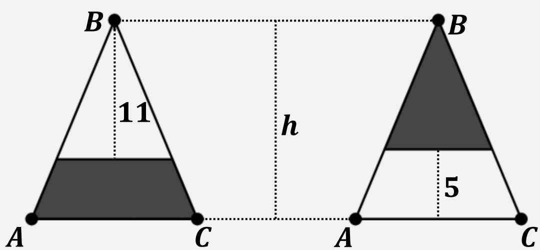
Mutually Exclusive Pattern Events in a Grid

PROBLEM NO.
076

DIFFICULTY

2023 AMC 8 #24

Isosceles $\triangle ABC$ has sides AB and BC of equal length. In the figure, lines are drawn parallel to AC so that the shaded regions of $\triangle ABC$ have equal areas. The heights of the two unshaded regions are 11 units and 5 units, respectively.



What is the total height h of $\triangle ABC$?

STEP 1 **Concept**

In isosceles $\triangle ABC$, a segment parallel to AC cuts off a small similar triangle at the top with height 11 units.

If the entire triangle has height h units, what fraction of the total area is this small top triangle?

STEP 2 Discovery

If the shaded regions in the two diagrams have equal area, which equation must be true?

A $\left(\frac{11}{h}\right)^2 = \left(\frac{h-5}{h}\right)^2$

B $1 - \frac{11}{h} = \frac{h-5}{h}$

C $\frac{11}{h} = \frac{h-5}{h}$

D $1 - \left(\frac{11}{h}\right)^2 = \frac{h-5}{h}$

E $1 - \left(\frac{11}{h}\right)^2 = \left(\frac{h-5}{h}\right)^2$

STEP 3 Application

If the shaded regions in the two diagrams have equal area, what is the total height h of $\triangle ABC$?

Planar Geometry Properties

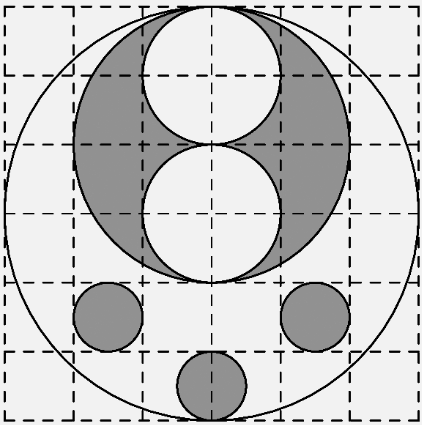
Composite Circle Areas

PROBLEM NO.
080

DIFFICULTY

2023 AMC 8 #12

Inside the large white circle shown below are several smaller circles, some shaded and some unshaded.



What fraction of the large circle's area is shaded?

STEP 1 Concept

Let R_L be the radius of the large outer circle, R_M the radius of the single middle circle, R_W the radius of each of the two white circles inside the middle circle, and R_S the radius of each of the three small shaded circles on the right.

Which tuple $(R_L,\ R_M,\ R_W,\ R_S)$ is correct?

A $\left(3, 2, 1, \frac{1}{2}\right)$

B $\left(3, \frac{5}{2}, 1, \frac{1}{2}\right)$

C $\left(\frac{7}{2}, 2, 1, \frac{1}{2}\right)$

D $\left(3, 2, \frac{1}{2}, \frac{1}{4}\right)$

E $\left(\frac{7}{2}, \frac{5}{2}, 1, \frac{1}{2}\right)$

STEP 2 Discovery

Using the result of Step 1,

what is the total area of all shaded regions inside the large circle, expressed as $k\pi$ square units (i.e., find k so that shaded area $= k\pi$)?

STEP 3 Application

What fraction of the large circle's area is shaded?

Solid Geometry Structure

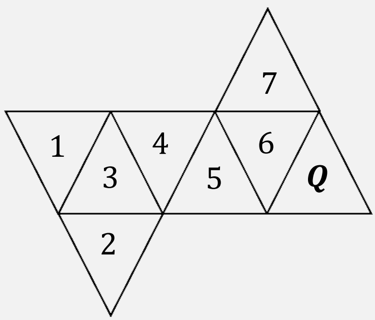
Nets and Spatial Relationships of Polyhedral

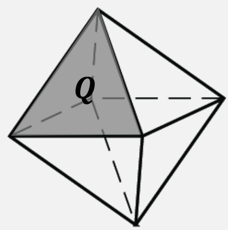
PROBLEM NO.
085

DIFFICULTY

2023 AMC 8 #17

A regular octahedron has 8 equal triangle faces, and 4 faces meet at each corner. The paper on the left is a net for this shape. Alex will fold it to make the octahedron shown on the right.





After folding, which numbered face will be right next to Q on its right side?

STEP 1 **Concept**

In the net, which numbered face is directly to the left of face Q?

STEP 2 Discovery

In the net of the regular octahedron from Step 1, the eight faces can be divided into an upper half and a lower half when the shape is folded.

Which of the following sets lists all of the faces that belong to the lower half of the octahedron?

A $1, 2, 3, 4$

B $2, 3, 4, 5$

C $3, 4, 5, 6$

D $2, 4, 5, 7$

E $4, 5, 6, 7$

STEP 3 Application

When the net is folded into a regular octahedron, which numbered face is directly to the right of face Q?

Solid Geometry Structure

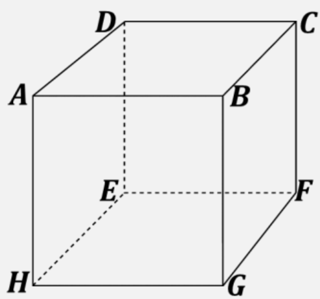
Cube Face Diagonals and Equilateral Triangle

PROBLEM NO.
092

DIFFICULTY
■■■■□

2024 AMC 8 #20

Any three vertices of the cube ABCDEFGH, shown in the figure below, can be connected to form a triangle. (For example, vertices A, B, and C can be connected to form isosceles $\triangle ABC$.)



How many of these triangles are equilateral and contain A as a vertex?

STEP 1 **Concept**

Complete the statement.

An equilateral triangle in a cube cannot be formed using only ____ⓐ___ because adjacent edges always meet at a 90° angle. Instead, it is formed when all three sides are ____ⓑ____ of square faces.

STEP 2 **Discovery**

List all face diagonals that have vertex A as one endpoint.

STEP 3 **Application**

Using the results from Steps 1 and 2,

how many equilateral triangles contain vertex A

Applied Geometry

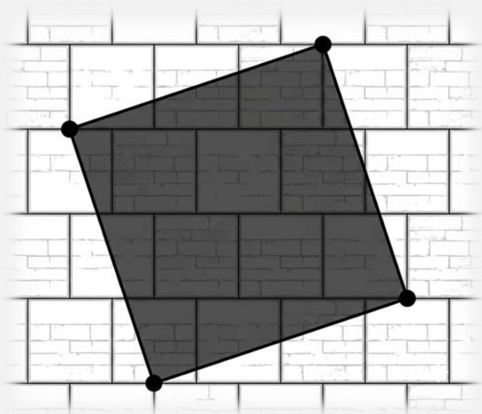
Pythagorean Theorem and Area of a Square

PROBLEM NO.
098

DIFFICULTY

2026 AMC 8 #13

The diagram below shows a pattern made from 1×1 unit squares. Each row of unit squares is moved horizontally by half a unit compared with the row directly above it. A shaded square is drawn over the pattern. Each corner of the shaded square is also a corner of one of the unit squares.



In square units, what is the area of the shaded square?

STEP 1 Concept

A right triangle has legs of lengths a and b, and a hypotenuse of length c.

Write the equation that relates a, b, and c.

STEP 2 Application

The diagram below shows a pattern made from 1×1 unit squares. Each row of unit squares is shifted horizontally by half a unit relative to the row directly above it. A shaded square is drawn over the pattern. Each vertex of the shaded square is also a vertex of one of the unit squares.

ⓐ One side of the shaded square is the hypotenuse of a right triangle.

What are the lengths of the two legs of this right triangle?

ⓑ Find the length of one side of the shaded square.

ⓒ Find the area of the shaded square.

Applied Geometry

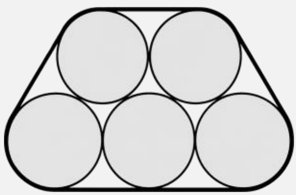
Arc Length and Tangent Geometry

PROBLEM NO.
113

DIFFICULTY

2026 AMC 8 #23

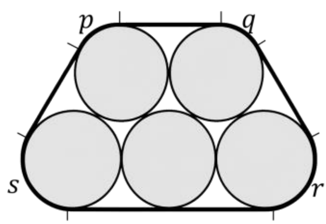
Nora has 5 identical round tokens, each with a diameter of 4 inches. She places the tokens in 2 rows on a tabletop, as shown below, and wraps a rubber band tightly around them.



Express the length of the rubber band in the form $a\pi + b$, where a and b are integers.

STEP 1 **Concept**

The four curved parts of the rubber band are labeled p, q, r, and s, as shown.



Write the central angle of each arc.

arc p = ________

arc q = ________

arc r = ________

arc s = ________

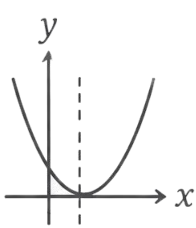
STEP 2 Discovery

Find the total length of the curved part of the rubber band.

STEP 3 Application

ⓐ Find the total length of the straight parts.

ⓑ Express the total length of the rubber band in the form $a\pi + b$.



$$a^2 + b^2 = c^2$$

Applying the formula for the area of a trapezoid,

$$area = \frac{1}{2} \times (1 + 3) \times 4 = 8,$$

we find the area of this trapezoid to be **8 square units**. Since the original question asks for **the area of the shaded region**, which is the **trapezoid area minus the area of the overlapping squares**, we **subtract** the area of the smallest square, **2** square units, **resulting in:** $\mathbf{8 - 2 = 6.}$

The final step is to compare this area to the **combined area of the three squares**. Given **each square** has an area of **4 square units**, their total area is; **4+4+4=12 square units**. Thus, the ratio of the shaded area to the total area of the squares is **6:12**, simplifying to **1:2**.

Therefore, the correct answer is **1:2** or $\frac{1}{2}$.

109. *Answer and Explanation*

Step 1:
Since the **regular hexagons are similar**, we know that their areas are proportional to the squares of their side lengths. The ratio of the **side lengths is 3:2**, so the ratio of the areas of the larger and smaller hexagons **is** $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$.
This means that if the area of smaller hexagon is A, then the **area of larger hexagon is** $\frac{9}{4}$**A**.

Step 2:
The combined area of the six trapezoids is the **difference between the larger and smaller triangle areas**, so the area of the six trapezoids is $\frac{9}{4}$**A − A =** $\frac{5}{4}$**A**.

Step 3:
Since the **six trapezoids are congruent**, the area of **one trapezoid is** $\frac{5}{4}$**A**$\times \frac{1}{6} = \frac{5}{24}$**A**. So, the r**atio** of the area of one trapezoid to the area of the inner hexagon is $\frac{5}{24}$**A: A=5:24**.

So, the ratio is $\mathbf{5:24}$.

Therefore, the correct answer is **5:24**.

110. ***Answer and Explanation***

Step 1.
① tile contribution (length, in inches):
There are $\boldsymbol{m}$ **tiles**, each of **side length** $\boldsymbol{s}$, so the tile portion along a side totals $\boldsymbol{ms}$ inches.

② frame contribution (length, in inches):
Between adjacent tiles there are $\boldsymbol{m-1}$ **shared gaps**, each **filled by one frame strip of width** $\boldsymbol{k}$. There are also two outer edges (left and right), each adding one more frame strip of width k. Hence the number of frame strips along **the side is** $\boldsymbol{(m-1)+2=m+1}$, contributing $\boldsymbol{(m+1)k}$ inches.

So, the total side length is $\boldsymbol{ms+(m+1)k}$.

Therefore, the correct answer is $\boldsymbol{ms+(m+1)k}$.

Step 2.
① tile area:
There are $\boldsymbol{m^2}$ **tiles**, each of **area** $\boldsymbol{s^2}$. Grouped in an $m\times m$ array, the total tile area is $(\boldsymbol{ms})^2$.

② park area:
The park is a square of **side** $\boldsymbol{ms+(m+1)k}$, so its total area is

$$\boldsymbol{(ms+(m+1)k)^2}.$$

Fraction covered by tiles:

$$\frac{\text{tile area}}{\text{park area}}=\frac{(\boldsymbol{ms})^2}{(\boldsymbol{ms}+(\boldsymbol{m}+\boldsymbol{1})\boldsymbol{k})^2}=\frac{\boldsymbol{64}}{\boldsymbol{100}}.$$

Therefore, the correct answer is **B**.

Step 3.
Since all lengths are positive, take the positive square root:

$$\frac{(ms)^2}{(ms+(m+1)k)^2}=\frac{64}{100}\quad\to\quad\frac{\boldsymbol{ms}}{(\boldsymbol{ms}+(\boldsymbol{m}+\boldsymbol{1})\boldsymbol{k})}=\frac{\boldsymbol{4}}{\boldsymbol{5}}.$$

So,

$$5ms=4(ms+(m+1)k)$$

and

$$5(24)s=4\big((24s)+(24+1)k\big)\quad\to\quad\boldsymbol{120s=96s+100k}$$
$$\to\quad 24s=100k.$$

Thus,

$$\frac{\boldsymbol{k}}{\boldsymbol{s}}=\frac{\boldsymbol{24}}{\boldsymbol{100}}=\frac{\boldsymbol{6}}{\boldsymbol{25}}.$$

Therefore, the correct answer is $\boldsymbol{\frac{24}{100}}$ or $\boldsymbol{\frac{6}{25}}$.

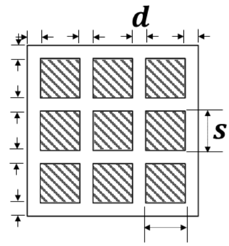
111. ***Answer and Explanation***

First, the area of one gray tile is;

$$\boldsymbol{s} \times \boldsymbol{s} = s^2.$$

When n=30, there are 900 tiles in total. So, **the total area of the tiles** is:$900 \times s^2 = \mathbf{900}\boldsymbol{s}^{\mathbf{2}}$.

To determine the side length of the large square, we need to consider the arrangement of the tiles and the border. When **n=30**, there are **29 borders between the 30 tiles**. Additionally, there are borders on both ends (left of the **first** tile and right of the **last** tile), making a total of 31 borders. Hence, the **total length of the borders** is $\mathbf{31}\boldsymbol{d}$.



The **side length of the large square** includes the **total length of the tiles plus the total length of the borders**:

side length of the large square=$\mathbf{30}\boldsymbol{s} + \mathbf{31}\boldsymbol{d}$

The area of the large square is the square of its side length:

$$(\mathbf{30}\boldsymbol{s} + \mathbf{31}\boldsymbol{d})^{\mathbf{2}}$$

According to the problem, the gray tiles cover 81% of the area of the large square. This can be expressed as:

$$\frac{900s^2}{(30s + 31d)^2} = 0.81 = \frac{81}{100}$$

Taking the square root of both sides, we get:

$$\frac{\mathbf{30}\ \boldsymbol{s}}{(\mathbf{30}\boldsymbol{s} + \mathbf{31}\boldsymbol{d})} = \frac{\sqrt{\mathbf{81}}}{\sqrt{\mathbf{100}}} \quad \rightarrow \quad \frac{30s}{30s + 31d} = \frac{9}{10}$$

$$300s = 9(30s + 31d) \;\; \rightarrow \;\; \mathbf{300}\boldsymbol{s} = \mathbf{270}\boldsymbol{s} + \mathbf{279}\boldsymbol{d}.$$

So,

$$\mathbf{279}\boldsymbol{d} = \mathbf{30}\boldsymbol{s} \quad \rightarrow \quad \frac{d}{s} = \frac{\mathbf{30}}{\mathbf{279}}.$$

Therefore, the correct answer is $\frac{\mathbf{30}}{\mathbf{279}}$

112. ***Answer and Explanation***

Step 1.

The length of a **quarter-circle arc** is always **one-fourth of the circumference of the full circle**.

A full circle with radius r has **circumference** $\mathbf{2}\boldsymbol{\pi r}$.
A quarter circle is one-fourth of a full circle, so its arc length is:

$$\frac{\mathbf{1}}{\mathbf{4}} \times 2\pi r = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\pi r}.$$

So,

$$\frac{1}{2}\pi(\mathbf{8}) = \boldsymbol{4\pi}.$$

Therefore, the correct answer is $\boldsymbol{4\pi}$.

Step 2.

When several quarter-circle arcs are connected, add their radii first, then **multiply by** $\boldsymbol{\frac{\pi}{2}}$.

Each quarter-circle arc has length $\boldsymbol{\frac{1}{2}\pi r}$.

The five radii are 1, 1, 2, 3, 5.

Their sum is $\mathbf{1+1+2+3+5=12}$.

So, the total length of the curve is:

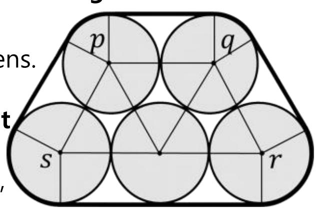
$$\frac{1}{2}\pi(\mathbf{12}) = \boldsymbol{6\pi}.$$

Therefore, the correct answer is $\boldsymbol{6\pi}$.

113. ***Answer and Explanation***

Step 1.

Use **equilateral triangles**, **rectangles**, and the **angle sum around a point** to find each central angle.



Connect the centers of the touching tokens. Since the tokens are congruent and touching, the **distances between adjacent centers are equal**.

So, the centers form **equilateral triangles**, so each triangle angle is $\mathbf{60}$°.

Next, **draw radii from the centers to the points** where the rubber band touches the tokens. A **radius to a tangent point is perpendicular** to the tangent line. These radii and the adjacent centers **form rectangles**, so the related angles are $\mathbf{90}$°.

For **arc** $\boldsymbol{p}$, the angles around the center add to 360°. The other angles are 60°, 60°, 90°, 90°.

So, $\boldsymbol{p} = 360° - 60° - 60° - 90° - 90° = \mathbf{60}°$.

By symmetry, $\boldsymbol{q} = \mathbf{60}°$.

For **arcs** $\boldsymbol{r}$ and $\boldsymbol{s}$, the other angles are 90°, 60°, 90°.

So $r = s = 360° - 90° - 60° - 90° = 120°$.

Therefore, the correct answer is **p=60°, q=60°, r=120°, s=120°**.

Step 2.

The **length of an arc** is determined by the fraction of the circle represented **by its central angle**.

An **arc length** with central angle θ is:

$$\textbf{Arc Length} = \textbf{Circumference} \times \frac{\boldsymbol{\theta}}{\mathbf{360^\circ}}.$$

Each token has a **diameter of 4** inches, so its **circumference is** $\boldsymbol{4\pi}$.
From Step 1,

$$\textbf{arc}\ \boldsymbol{p} = 4\pi \times \frac{\mathbf{60^\circ}}{360^\circ} = \frac{\boldsymbol{2\pi}}{\mathbf{3}},$$

$$\text{arc}\ q = 4\pi \times \frac{60^\circ}{360^\circ} = \frac{2\pi}{3},$$

$$\text{arc}\ r = 4\pi \times \frac{120^\circ}{360^\circ} = \frac{4\pi}{3},$$

$$\text{arc}\ s = 4\pi \times \frac{120^\circ}{360^\circ} = \frac{4\pi}{3}.$$

So,

$$\frac{\boldsymbol{2\pi}}{\mathbf{3}} + \frac{\boldsymbol{2\pi}}{\mathbf{3}} + \frac{\boldsymbol{4\pi}}{\mathbf{3}} + \frac{\boldsymbol{4\pi}}{\mathbf{3}} = \boldsymbol{4\pi}.$$

Therefore, the correct answer is $\boldsymbol{4\pi}$.

Step 3.

ⓐ The total length of the straight part is the sum of all straight segments.

The rubber band has **four straight segments**. Their lengths are **4**, **4**, **8**, and **4**. The total straight length is found by adding these four segments. Then **4+8+4+4=20**.

Therefore, the correct answer is **20**.

ⓑ The total length of the rubber band is the sum of its straight part and its curved part.

From Step 3 ⓐ, the total length of the **straight part is 20**.
From Step 2, the total length of the **curved part is** $\boldsymbol{4\pi}$.
Adding these two parts gives the total length of the rubber band.
Then: $20 + 4\pi = \boldsymbol{4\pi + 20}$.

Therefore, the correct answer is $\boldsymbol{4\pi + 20}$.