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Book 4: **COUNTING and PROBABILITY**

Combinatorics

Permutations with Adjacency Restrictions

PROBLEM NO. **118**

DIFFICULTY ■■■□□

2020 AMC 8 #10

Liam owns a set of 4 toy cars: a Red Racer, a Blue Cruiser, a Silver Bullet, and a Golden Beast. He wants to line them up in a single row for display, but he does not want the Silver Bullet and the Golden Beast to be placed right next to each other.

In how many different ways can he arrange them?

STEP 1 Concept

Liam has 4 distinct toy cars: a Red Racer, a Blue Cruiser, a Silver Bullet, and a Golden Beast.

If there is no restriction, in how many different ways can he arrange the four cars in a single row?

STEP 2 Discovery

Liam has 4 distinct toy cars: Red Racer (R), Blue Cruiser (B), Silver Bullet (S), and Golden Beast (G).

If S and G must be placed next to each other in the row, in how many different arrangements can he line up the four cars?

A 6

B 8

C 10

D 12

E 18

STEP 3 Application

He wants to display them in a single row so that S and G are not next to each other.

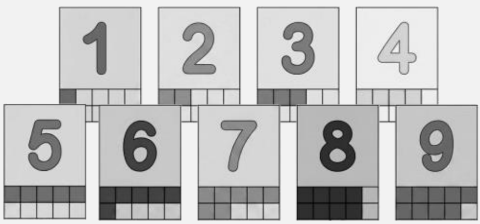
How many different arrangements are possible?

Combinatorics

Equal-Sum Set Partitions

PROBLEM NO. **126** | DIFFICULTY ■■■■□ | 2023 AMC 8 #21

Sofia writes the numbers 1, 2, ..., 9 on separate cards, with one number on each card. She wants to split the cards into 3 groups of 3 cards each so that the sum of the numbers in every group is the same.



How many different ways can she do this?

STEP 1 **Concept**

If the sum of the numbers in each group must be the same, what should that sum be?

STEP 2 Discovery

If the card numbered 9 is placed in one group, which pairs of numbers can be chosen with it so that the group's sum is the required total from Step 1?

A (1, 4) and (2, 3)

B (1, 6) and (2, 5)

C (1, 6) and (2, 4)

D (2, 4) and (1, 5)

E (3, 5) and (2, 6)

STEP 3 Application

Using the result of Step 1 and Step 2,

how many different ways can she do this?

General Counting

Counting Arrangements with Repeated Objects

PROBLEM NO. **135** | DIFFICULTY ■■■■■ | 2026 AMC 8 #20

The town of Riverton uses gold coins and silver coins. Gold coins are 1 mm thick, and silver coins are 3 mm thick.

In how many different ways can Nora make a stack of coins that is 8 mm tall using gold and silver coins, if the order of the coins matters?

STEP 1 Concept

A gold coin is 1 mm thick, and a silver coin is 3 mm thick. A stack contains 2 gold coins and 1 silver coin.

What is the total height of the stack?

STEP 2 Discovery

A stack contains 3 gold coins and 1 silver coin.

How many different stacks are possible?

STEP 3 Application

A stack is 8 mm tall.

ⓐ Find all possible (gold coins, silver coins) pairs that make a total height of 8 mm.

ⓑ For each pair, find the number of different stacks.

ⓒ Find the total number of different stacks.

General Counting

Two-Way Table Reasoning

PROBLEM NO. 140 | DIFFICULTY ■■■■□ | 2024 AMC 8 #19

Avery has a collection of 15 pairs of sneakers. Three-fifths of the sneakers are red, while the rest are white. Out of all the pairs, two-thirds are high-top sneakers, and the rest are low-top. The red high-top sneakers represent a fraction of the entire collection.

What is the minimum possible value of this fraction?

STEP 1 Concept

Avery owns 15 pairs of sneakers:

three-fifths of the pairs are red,

two-thirds of the pairs are high-top.

Complete the table.

Category	Number of Pairs
Red	
White	
High-Top	
Low-Top	

STEP 2 **Discovery**

Use your completed table from Step 1.

Complete the two-way table.

	High-Top	Low-Top	Total
Red			
White			
Total			**15**

STEP 3 **Application**

Use your completed tables from Steps 1 and 2.

What fraction of Avery's 15 pairs are red high-top?

General Counting

Grid Coverage Optimization

PROBLEM NO. 142 | DIFFICULTY ■■■■■ | 2024 AMC 8 #16

Leo fills a 9×9 board with the integers from 1 to 81 in any arrangement. For each row and each column, he finds the product of its numbers.

What is the least number of rows and columns that could have a product divisible by 3?

STEP 1 **Concept**

ⓐ If none of the integers in a row is divisible by 3, can the product of the integers in that row be divisible by 3?

ⓑ How many multiples of 3 are there among the integers from 1 to 81?

STEP 2 Discovery

Write each answer using the variables r and c, if needed.

Suppose the 27 multiples of 3 occupy exactly r rows and c columns of a 9×9 grid.

ⓐ How many cells are at the intersections of these r rows and c columns?

ⓑ Since all 27 multiples of 3 must be in these intersection cells, what inequality must r and c satisfy?

STEP 3 Application

Use your answers from Steps 1 and 2.

ⓐ If $r + c = 10$, what is the greatest possible value of rc?

ⓑ Can 27 multiples of 3 fit in the intersection cells when $r + c = 10$?

ⓒ What is the least possible value of $r + c$?

ⓓ The integers from 1 to 81 are placed in a 9×9 grid, with one integer in each cell.

For each row and each column, the nine integers in it are multiplied together.

What is the least possible sum of the number of row products divisible by 3 and the number of column products divisible by 3?

Probability

Compound Probability with a Number Property

PROBLEM NO. 145 | DIFFICULTY ■■■□□ | 2022 AMC 8 #12

Suppose Spinner A and Spinner B are spun at the same time. Form a number N by multiplying the outcome on Spinner A by 10 and then adding the outcome on Spinner B.

What is the probability that this number N is a perfect square?

STEP 1 Concept

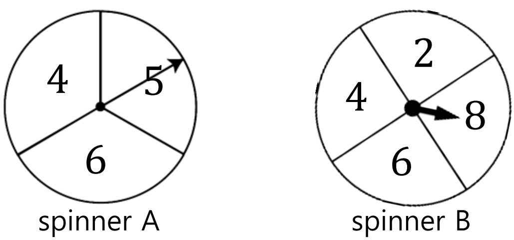
How many possible results $(A,\ B)$ can be obtained from spinning the two spinners?

STEP 2 Discovery

What is the probability that N is a perfect square?

STEP 3 Application

Spinner A has the numbers 4, 5, 6. Spinner B has the numbers 2, 4, 6, 8. Both spinners are spun once and N is the product of the two numbers A and B.



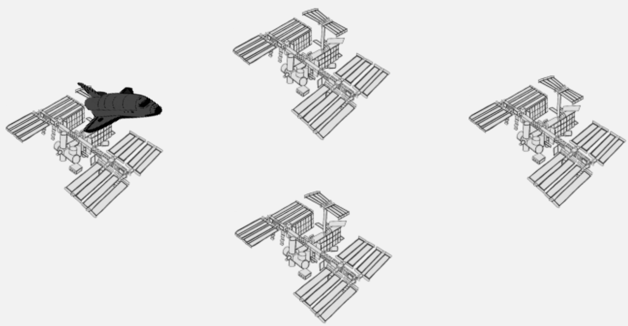
What is the probability that N is a perfect square?

Probability

Volume of a Prism from Its Net

PROBLEM NO. 147 | DIFFICULTY | 2022 AMC 8 #25

A space probe navigates among 4 orbiting stations. At every move, it must dock at one of the other 3 stations, each with equal chance.



After completing exactly 4 moves, what is the probability that the probe has returned to its original station?

STEP 1 Concept

A space probe is currently at its starting station S and moves among 4 orbiting stations. On each move, it must dock at one of the other 3 stations, all equally likely.

If the probe is at some station different from S right now, what is the probability that its very next move lands it at S?

STEP 2 Discovery

A space probe starts at station S and moves among 4 stations. On each move, it must dock at one of the other 3 stations, all equally likely.

After exactly 2 moves, what is the probability that the probe is back at station S?

A 0

B $\frac{1}{9}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

STEP 3 Application

A space probe starts at station S and moves among 4 stations. On each move, it must dock at one of the other 3 stations, all equally likely.

After exactly 4 moves, what is the probability that the probe is back at station S?

Step 3.

① start with the first possible group containing 9
From Step 2, if we choose **(9,2,4)**:
→ remaining cards: {1,3,5,6,7,8}.
Pick the largest remaining number **(8)** for the second group.
To make 15: 15−8=7 → need (1,6) → second group **(8,1,6)**
Remaining cards: **{3,5,7}**, sum = 15 → valid third group.

② try the second possible group containing 9
If we choose **(9,1,5)**: → remaining cards: {2,3,4,6,7,8}.
Pick the largest remaining number **(8)** for the second group.
To make 15: 15−8=7 → need (3,4). → second group: **(8,3,4)**.
Remaining cards: **{2,6,7}**, sum = 15 → valid third group.

Count distinct arrangements
arrangement 1: (9,2,4), (8,1,6), (7,3,5),
arrangement 2: (9,1,5), (8,3,4), (7,2,6).

There are **exactly 2 valid ways** to form the groups.

Therefore, the correct answer is **2**.

127. *Answer and Explanation*

Find the total sum of all the elements:
The **sum of numbers from 1 to 12** can be calculated,

$$1+2+\cdots+11+12=\frac{\mathbf{12\times(1+12)}}{\mathbf{2}}=\mathbf{78}.$$

Divide the total sum by the number of groups:
Since we want **to divide the cards into 3 groups**, we divide the total sum by 3 to get the sum of each group. So, sum of each group is $\frac{\mathbf{78}}{\mathbf{3}}$ **=26**.

To make counting easier, we can consider the possible groups that the number 12. The possible groups are (12, 1, 2, 11), (12, 1, 3, 10), (12, 1, 4, 9), (12, 1, 5, 8), (12, 2, 3, 9), (12, 2, 4, 8), (12, 2, 5, 7), (12, 3, 4, 7) and (12, 3, 5, 6). We can eliminate (12, 1, 2, 11) and (12, 1, 3, 10) which are symmetrical.

We can do exhaustive search on the remaining cases as there are just few of them and by eliminating symmetry we will proceed faster. By checking all cases, we will find out that only (12, 1, 5, 8), (12, 2, 5, 7), and (12, 3, 4, 7) lead to the correct splitting.

Considering symmetry (by replacing 1 with 2), the correct group splitting is
(12, 1, 5, 8) (11, 2, 6, 7) (10, 3, 4, 9) and
(12, 2, 5, 7) (11, 1, 6, 8) (10, 3, 4, 9) and
(12, 3, 4, 7) (11, 1, 5, 9) (10, 2, 6, 8).

So, there are **3 ways** to arrange the cards.

Therefore, the correct answer is **3**.

128. *Answer and Explanation*

Step 1.

A number is a **multiple of 15** exactly when it is a **multiple of both 3 and 5**, because 15=3×5 and **3 and 5 are coprime**.
divisible by 5 ⇔ the last digit is **0** or **5**,
divisible by 3 ⇔ the **sum of the digits is a multiple of 3**.

So, to be divisible by 15, both conditions must hold simultaneously.

Therefore, the correct answer is **E**.

Step 2.

From Step 1, a number divisible by 5 must end with **0 or 5**.

Because N is a zigzag number of **length 5**, its **1st, 3rd, and 5th** digits are **equal**, and its **2nd and 4th digits are equal** (but different from the others). So, whatever the last digit is, the first and third digits must match it.

If the last digit were 0, then the **first** digit would **also be 0**. But a five-digit integer **cannot start with 0**. Hence the last digit cannot be 0. So the **last digit must be 5**. By the **alternating pattern**, positions **1, 3, and 5 are all 5**, while positions 2 and 4 are the same but different from 5.

This forces the **form $5x5x5$** with $x \neq 5$.

Therefore, the correct answer is **C**.

Step 3.

From Step 2, any five-digit zigzag number divisible by 5 must be of the form **$5x5x5$** with $x \neq 5$. To be a multiple of 15, the number must **also be a multiple of 3**.

For $5x5x5$, the **sum of digits** is: $\mathbf{5+x+5+x+5=15+2x}$.

A number is divisible by 3 if its **digit sum is divisible by 3**. Since 15 is a multiple of 3, we need

$$\mathbf{15+2x\equiv 0\ (mod\ 3)} \quad \rightarrow \quad \mathbf{2x \equiv 0\ (mod\ 3)}.$$

Because gcd(2,3)=1, this **holds exactly when $x\equiv 0 \pmod 3$**. Among the digits 0,1,...,9, the multiples of 3 are **0, 3, 6, 9**. All of these are allowed (they are digits and **each is $\neq 5$**), and none causes a leading-zero issue since the first digit is 5.

Thus the four valid choices of x yield four valid numbers:

$$\mathbf{x=0,\ 3,\ 6,\ 9} \quad \rightarrow \quad \mathbf{50505},\ \mathbf{53535},\ \mathbf{56565},\ \mathbf{59595}.$$

So there are **4 such numbers**.

Therefore, the correct answer is **4** or **four**.

129. *Answer and Explanation*

Step 1.

When a grid is folded in half, **every two matching unit squares form one overlapping pair**.

A 6-by-6 grid has **6 rows** and **6 columns**.
After folding vertically, each row forms 3 overlapping pairs.
So, **6×3=18**.

Therefore, the correct answer is **18**.

Step 2.

A **red-on-red pair is possible only** when **neither square in the pair is blue**.

So,
  more blue-on-blue pairs,
  fewer pairs containing **only one blue square**,
means
  more red-on-red pairs.

Comparing the two figures, **Figure B has more blue-on-blue pairs**.
So **Figure B has more red-on-red pairs**.

Complete the sentence.
The more blue squares overlap after folding, the **greater** the number of red-on-red pairs.

Therefore, the correct answer is **greater** or **larger** etc.

Step 3.

There are **18 overlapping pairs in total**.
A pair is **not a red-on-red pair** if it contains **at least one blue square**.
So,
spread the blue squares apart to minimize the number of red-on-red pairs,
overlap the blue squares as much as possible to maximize the number of red-on-red pairs.

ⓐ There are **18 overlapping pairs**.

Therefore, the correct answer is **18**.

ⓑ To make the number of red-on-red pairs **as small as possible**, place the **13 blue squares in 13 different pairs**.
Then, **18−13=5**.

Therefore, the correct answer is **5**.

ⓒ To make the number of red-on-red pairs **as large as possible**, overlap the blue squares whenever possible.
Since **13 blue squares** require at least
6 blue-on-blue pairs and 1 additional pair,
the **blue squares occupy 7 pairs**.

Thus, **18−7=11**.

Therefore, the correct answer is **11**.

ⓓ
So, $\boldsymbol{m+M=5+11=16}$.

Therefore, the correct answer is **16**.

130. *Answer and Explanation*

Step 1.

A **combination** is used whenever the **order of selection does not matter**.

For example, **choosing seats A, B, C, and D** represents the same group of empty seats as **choosing D, C, B, and A**. Although the order is different, the selected seats are **exactly the same**.
So, these should be **counted as one selection**, not different ones.
The number of ways to **choose r objects from n objects** is given by the combination formula:

$$\binom{n}{r}=\frac{n!}{r!\,(n-r)!}.$$

In this problem, we are not assigning passengers to seats. Instead, we are simply **choosing which 4 of the 12 seats are empty**.
Using the combination formula,

$$\binom{\mathbf{12}}{\mathbf{4}} = \frac{12!}{4!\,(12-4)!} = \frac{\mathbf{12}!}{\mathbf{4}! \times \mathbf{8}!} = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = \mathbf{495}.$$

These 495 combinations represent **all possible sets of 4 empty seats**. Since every set of 4 empty seats is equally likely, **495 is the total number of possible outcomes** and will be used as the denominator when calculating the probability in the next step.

Therefore, the correct answer is **495**.

Step 2.

We need the **four empty seats to contain at least one pair of adjacent empty seats** in the same row. Instead of focusing on where the empty seats are located, classify the arrangements by how the empty seats are connected.

Use the following notation:

- **P** = **one pair of adjacent empty seats**,
- **S** = **one single empty seat**,
- **E = one entire empty row** (3 consecutive empty seats).

Moving the same pattern to a different row does not create a new pattern because the adjacency relationship remains the same.

There are **only three possible empty-seat patterns**.

Pattern 1: **1P+2S**.
There is **exactly one pair of adjacent empty seats**, and the other **two empty seats are single empty seats**.

Pattern 2: **1E+1S**.
There is **one entire empty row**, and the **remaining one empty seat** is in a different row.

Pattern 3: **2P**.
There are **two separate pairs of adjacent empty seats**, with one pair in each of two different rows.

Every successful arrangement must contain at least one adjacent pair of empty seats in the same row. Any arrangement satisfying the condition must match exactly one of the three patterns above.

Therefore, the correct answer is **3** or **three**.