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Book 5: **THINKING SKILLS**

Logical Problem

Tournament Results Deduction

PROBLEM NO. **153**

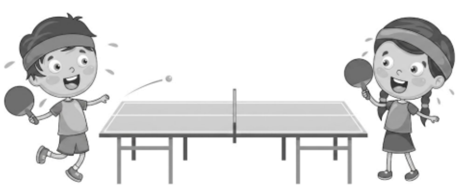
DIFFICULTY ■■■■□

2023 AMC 8 #08

Mina, Momo, Rika, and Riku took part in a table tennis league play. Each player played against each of the other three players exactly twice. The table below shows the win–loss records for the players. The numbers 1 and 0 indicate a win or a loss, respectively. For instance, Mina won five matches and lost the fourth match.

player	result
Mina	111011
Momo	101010
Rika	010100
Riku	??????

What was Riku's win–loss record?



STEP 1 Concept

How many games were played in total during the league play?

STEP 2 Discovery

Using the result of Step 1, how many games did Riku win?

STEP 3 Application

What was Riku's win-loss record?

A 0 0 0 1 0 1

B 0 1 0 0 0 1

C 1 0 0 1 0 0

D 0 0 1 0 1 0

E 1 0 0 0 1 0

Logical Problem

Linear Seating with Relative-Position Constraints

PROBLEM NO.
154

DIFFICULTY
■■■■□

2020 AMC 8 #06

Ethan, Lucas, Mia, Chloe, and Zoe boarded a ride with five seats in a single row, one passenger per seat. Chloe sat in the last seat. Ethan sat right behind Zoe. Lucas sat in one of the seats in front of Ethan. At least one passenger sat between Mia and Lucas.

Who sat in the middle seat?

STEP 1 Concept

Five friends — Ethan, Lucas, Mia, Chloe, and Zoe — are going to sit in a ride that has five seats in a single row, one passenger per seat. It is known that Chloe sits in the last seat (the seat at the very end of the row).

How many possible ways are there to arrange the remaining four friends in the other seats?

STEP 2 Discovery

In addition, Ethan sits directly behind Zoe.

How many different seating arrangements are possible?

A 4

B 6

C 8

D 10

E 12

STEP 3 Application

Who sat in the middle seat?

Modular Arithmetic

Congruence and Remainder Reasoning

PROBLEM NO.
168

DIFFICULTY
■■□□□

2024 AMC 8 #06

Mia lists the numbers 15, 16, 17, 18, 19. After she removes one of her numbers, the sum of the other four numbers is a multiple of 4.

Which number did she remove?

STEP 1 **Concept**

The notation $a \equiv b \pmod{4}$ means that a and b leave the same remainder when divided by 4.

Complete the statements using 0, 1, 2, or 3.

$15 \equiv \square \pmod{4}$, $16 \equiv \square \pmod{4}$, $17 \equiv \square \pmod{4}$,
$18 \equiv \square \pmod{4}$, $19 \equiv \square \pmod{4}$.

STEP 2 **Discovery**

The numbers 15, 16, 17, 18, and 19 are written on a sheet of paper.

Find the sum of the five numbers.
How much greater is the sum than the greatest multiple of 4 less than or equal to it?

STEP 3 **Application**

Mia lists the numbers 15, 16, 17, 18, and 19. After she removes one of her numbers, the sum of the other four numbers is a multiple of 4.

Which number did she remove?

Modular Arithmetic

Pairing Numbers with the Same Remainder Modulo 10

PROBLEM NO. 169 | DIFFICULTY ■■■□□ | 2024 AMC 8 #16

Five different integers from 1 through 10 are selected, and five different integers from 11 through 20 are selected. No two selected numbers differ by 10.

What is the sum of the ten selected numbers?

STEP 1 Concept

The numbers from 1 to 20 can be grouped into pairs that differ by 10. $(1, 11)$, $(2, 12)$, $(3, 13)$, … , $(10, 20)$.

If you may choose at most one number from each pair, what is the greatest number of integers you can choose?

STEP 2 Discovery

A student chooses one number from each pair below.

Pair	Chosen Number
(1, 11)	1
(2, 12)	12
(3, 13)	3
(4, 14)	14
(5, 15)	15

Which ones digits do you predict will appear from the remaining pairs?

STEP 3 Application

Five different integers from 1 through 10 are selected, and five different integers from 11 through 20 are selected. No two selected numbers differ by 10.

ⓐ Find the sum of the integers from 1 to 10.

ⓑ How much larger is the total if five of the selected numbers come from 11 through 20 instead of 1 through 10?

ⓒ Find the sum of the ten selected numbers

Modular Arithmetic

Cyclic Patterns of Powers

PROBLEM NO. 173

DIFFICULTY ■■■■□

AMC 8 Similar

What is the tens digit of 7^{2029}?

Today's Takeaway

When an exponent is very large, don't calculate the entire number.

Look for a repeating pattern in the last one or two digits. Big exponents are solved with patterns, not with long calculations.

Modular Arithmetic

Units Digit of Factorial-Type Sums

PROBLEM NO. 174

DIFFICULTY

2022 AMC 8 #17

When n is an even number greater than 0, the expression $n!!$ is defined as the product of every even number starting from 2 and ending at n. For instance, $8!!=2\times4\times6\times8$. Now consider the sum $2!!+4!!+6!!+\cdots+2032!!$.

What digit appears in the units digit of this sum?

STEP 1 **Concept**

Let n be an even integer with $10\le n\le 98$.

What is the units digit of $n!!$?

STEP 2 Discovery

What is the units digit of

$$2!!+4!!+6!!+8!!?$$

A 0

B 2

C 4

D 6

E 8

STEP 3 Application

Consider the sum $2!!+4!!+6!!+\cdots+2032!!$.

What is the units digit of this sum?

Modular Arithmetic

Divisibility Rules with Digit Arrangement

PROBLEM NO. **180** | DIFFICULTY ■■■■□ | AMC 8 Similar

Lucy arranged her collection of pebbles in a straight line so that consecutive pebbles were equally spaced. She labeled sixteen consecutive integers b_1, b_2, b_3, … , b_{16}, which form an arithmetic sequence.

Suppose

$$1 \le b_1 \le 10, \qquad 19 \le b_2 \le 28, \qquad 259 \le b_{16} \le 267.$$

What is the sum of the digits of b_{19}?

Common Mistake

A common mistake is to assume a possible common difference immediately without checking that it satisfies all three given conditions.

Data and Statistics

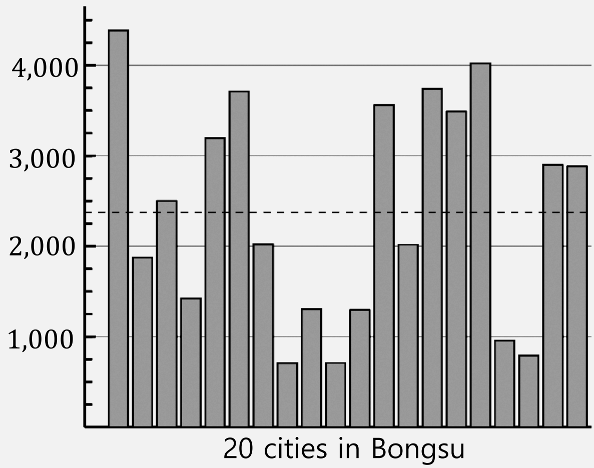
Mean from Graphical Data

PROBLEM NO.
182

DIFFICULTY

2020 AMC 8 #14

The county of Bonsu has 20 towns. The bar chart below shows the populations of these towns. The dashed horizontal line represents the average population of the 20 towns.



Based on the chart, which of the following is closest to the total population of all 20 towns?

A 40,000

B 42,500

C 45,000

D 47,500

E 50,000

Data and Statistics

Means of Overlapping Pairs

PROBLEM NO.
185

DIFFICULTY

2022 AMC 8 #16

Suppose four numbers are listed in sequence. The mean of the first two is 21, the mean of the next two is 26, and the mean of the final two is 30.

What is the mean of the first and last of the numbers?

STEP 1 **Concept**

What is the sum of the first two numbers?

A 18

B 21

C 40

D 42

E 44

STEP 2 Discovery

Let four numbers in order be a, b, c, d.

What is $a+d$?

A 44

B 48

C 50

D 52

E 56

STEP 3 Application

What is the mean of the first and last numbers?

Use results from Step 2, the **unit digit** of 2!!+4!!+6!!+8!! **is 2**.
So,

$$\begin{aligned}&2!! + 4!! + \cdots + 2032!!\\&= 2!! + 4!! + 6!! + 8!! + \mathbf{10}!! + \mathbf{12}!! + \cdots + \mathbf{2032}!!\\&\equiv 2 + \mathbf{0} \ (\text{mod } 10) = \mathbf{2}.\end{aligned}$$

Therefore, the correct answer is **2**.

175. *Answer and Explanation*

① identify when factorials become multiples of 15
We know **15=3×5**. **5!**=1×2×**3**×4×**5**=120, which is a multiple of 15. For any n≥5, the factorial n! will include the factors 3 and 5, so n! must also be a multiple of 15.
Thus, for all **n≥5**, **n!≡0 (mod15)**.

② reduce the original sum
Because every term from **5! to 50! is divisible by 15**, they **do not affect the remainder**. So the problem reduces to 1!+2!+3!+4!.
Adding them gives:

$$\mathbf{1}! + \mathbf{2}! + \mathbf{3}! + \mathbf{4}! = 1 + 2 + 6 + 24 = \mathbf{33}.$$

Divide 33 by 15:

$$33 \div 15 = 2 \dots \boldsymbol{R3}.$$

So the **remainder is 3**.

Therefore, the correct answer is **D**.

176. *Answer and Explanation*

The problem involves finding the smallest number of candies in a glass jar that satisfies two conditions when divided among a group of children.

Firstly, when the candies are **divided equally among 7 girls**, there are **3 candies left over**. This implies that the total number of candies is **3 more than a multiple of 7**.
However, considering the method of solving, it's understood that the effective number of candies is **actually 4 less than a multiple of 7** because adding 4 to the remainder makes it a complete multiple (3+4=7).

Secondly, when the candies are **divided equally among 6 boys**, **2 candies are left over**, suggesting that **the total number of candies** is **2 more than a multiple of 6**.
Similarly, by adding four to the remainder, the total becomes a number **4 less than a multiple of 6** (2+4=6).

To solve the puzzle, we look for the **Least Common Multiple (LCM) of 6 and 7**, which represents the smallest number that is a multiple of both six and seven. The LCM of 6 and 7 is **42**. Since **the number of candies is 4 less than this LCM** to satisfy both conditions, we subtract four from 42, resulting in

$$\mathbf{38 - 4 = 38},$$

candies in the jar.

When these 38 candies are divided among **13 children (the combined group of girls and boys**), the division results in;

$$\mathbf{38 \div 13 = 2 \ldots R8}$$

2 candies for each child with **8 candies remaining**.

Therefore, the correct answer is **8**.

177. ***Answer and Explanation***

Step 1.

① understanding the motion

Every **rightward leap** moves Noah **+5 posts** from his current position. Every **leftward leap** moves Noah **−3 posts** from his current position.

② expressing displacement mathematically

After $\boldsymbol{x}$ **rightward leaps**, the total forward displacement is $\mathbf{5x}$ posts. After $\boldsymbol{y}$ **leftward leaps**, the total backward displacement is $3y$ posts. Since these are **backward**, we subtract: $\mathbf{-3y}$.

The net displacement from the starting point is:

$$\textbf{net displacement}=\mathbf{5x-3y}$$

Therefore, the correct choice is **C**.

Step 2.

We know: $\mathbf{2023+3y}$ must be **divisible by 5**.
In other words:

$$\mathbf{2023 + 3y \equiv 0\ (mod\ 5).}$$

First, find 2023 mod 5:

2023÷5=404 **remainder 3**, so: $\mathbf{2023 \equiv 3\ (mod\ 5)}$.

Set up the modular equation

$$2023 + 3y \equiv 0 \text{ (mod 5)} \quad \rightarrow \quad \mathbf{3} + 3y \equiv \text{(mod 5)}.$$

We can subtract 3 from both sides:

$$\mathbf{3y \equiv -3\ (mod\ 5).}$$

Since $\mathbf{-3 \equiv 2\ (mod\ 5)}$, this is:

$$\mathbf{3y \equiv 2\ (mod\ 5).}$$

The **inverse of 3 modulo 5 is 2**:

$$\begin{aligned} \mathbf{3 \times 2 \equiv 1\ (mod\ 5)} \quad &\rightarrow \quad 3 \times 2 \times \mathbf{2} \equiv 1 \times \mathbf{2}\ (\text{mod } 5) \\ &\rightarrow \quad 3 \times \mathbf{4} \equiv \mathbf{2}\ (\text{mod } 5) \\ &\rightarrow \quad 3y \equiv 2\ (\text{mod } 5) \end{aligned}$$

The smallest positive y satisfying this is: $\boldsymbol{y}$**=4**.

Therefore, the correct answer is **4**.

Step 3.

From Step 2, y=4 leftward leaps are required so that 2023+3y is a multiple of 5.

Substitute $\boldsymbol{y}$**=4**: 2023+3×4=2035.
Thus, if Noah temporarily **overshoots to 2035 posts**, the total distance is divisible by 5.

Each rightward leap covers 5 posts, so to reach 2035 posts purely by rightward leaps:

$$\boldsymbol{x = \frac{2035}{5} = 407.}$$

From 2035, Noah **takes** $\boldsymbol{y}$**=4 leftward leaps** (each subtracting 3 posts) to land exactly at 2023 posts.

total leaps=$\boldsymbol{x}$**+**$\boldsymbol{y}$**=407+4=411**.

Therefore, the correct answer is **411**.

178. ***Answer and Explanation***

To solve the problem, we need to determine which digit corresponds to A in the six-digit number $ABCDEF$, where each of **the digits 1, 2, 3, 4, 5, and 6 is used exactly once**. The problem states that the four-digit number $ABCD$ is divisible by 8, the three-digit number BCD is divisible by 3, and the two-digit number EF is divisible by 9.

First, **consider the two-digit number** $\boldsymbol{EF}$, which must be **divisible by 9**. The possible values for $\boldsymbol{EF}$ using the digits 1 through 6 are **45 and 63**.

Next, examine the condition that the **three-digit number** $\boldsymbol{BCD}$ must be **divisible by 3**. For a number to be divisible by 3, the **sum of its digits must be a multiple of 3**.

If $\boldsymbol{EF}$ **is 45**, the digits E and F are 4 and 5, leaving the digits 1, 2, 3, and 6 for A, B, C, and D. We need to ensure that the **sum of** $\boldsymbol{B}$**,** $\boldsymbol{C}$**, and** $\boldsymbol{D}$ **is divisible by 3**. Using the digits 1, 2, 3, and 6, we find that $\boldsymbol{B+C+D}$ **= 1+2+3 = 6**, which is divisible by 3.

Finally, we check the condition that $ABCD$ must **be divisible by 8**. The last three digits of $ABCD$, which are $\boldsymbol{BCD}$**, should be divisible by 8**. With the remaining digits, the only combination that fits is **312**, because 312 is divisible by 8.

So, $\boldsymbol{A}$ **is** the remaining digit, which is **6**, making the six-digit number **631245**.

Therefore, the correct answer is **6**.

179. *Answer and Explanation*

To determine the total distance Benjamin drove over 20 days, we recognize that his driving distances form an **arithmetic sequence**. An arithmetic sequence is a sequence of numbers in which the difference between consecutive terms is constant.
Here, the **first term $\boldsymbol{a_1}$ is 2** km, and the **common difference $\boldsymbol{d}$ is 2** km.

The n-th term of an arithmetic sequence can be found using the formula: $\boldsymbol{a_n = a_1 + (n-1)d}$
For Benjamin, the 20th term (a_{20}) can be calculated as:

$$\boldsymbol{a_{20} = 2 + (20-1) \times 2 = 40}.$$

The sum of the first n terms of an arithmetic sequence (S_n) is given by:

$$\boldsymbol{S_n = \frac{n(a_1 + a_n)}{2}}.$$

Substituting the known values:

$$\boldsymbol{S_{20}} = \frac{20 \times (2+40)}{2} = \boldsymbol{420}.$$

So, the total distance Benjamin drove over 20 days is 420 km.

Therefore, the correct answer is **420**.

180. *Answer and Explanation*

Step 1: Find the possible values for the common difference.
The numbers given are individual terms of the arithmetic sequence, so $\boldsymbol{b_{20} = b_1 + (20-1)d = b_1 + 19d}$.
Minimum of common difference is $\boldsymbol{\frac{259-10}{19} = 13.105}$ ... and **maximum** of the value is $\boldsymbol{\frac{267-1}{19} = 14}$.
So, the **common difference is 14**.